2

THE RATIONAL NUMBERS

When *a* divided by *b* is not an integer, the quotient is a fraction. The Babylonians, who used a number system based on 60, expressed the quotients:

$$20 \div 8 \text{ as } 2 + \frac{30}{60} \text{ instead of } 2\frac{1}{2}$$

$$21 \div 8 \text{ as } 2 + \frac{37}{60} + \frac{30}{3,600} \text{ instead of } 2\frac{5}{8}$$

Note that this is similar to saying that 20 hours divided by 8 is 2 hours, 30 minutes and that 21 hours divided by 5 is 2 hours, 37 minutes, 30 seconds. This notation was also used by Leonardo of Pisa (1175–1250), also known as Fibonacci.

The base-ten number system used throughout the world today comes from both Hindu and Arabic mathematicians. One of the earliest applications of the base-ten system to fractions was given by Simon Stevin (1548–1620), who introduced to 16th-century Europe a method of writing decimal fractions. The decimal that we write as 3.147 was written by Stevin as 3 ⓪ 1 ① 4 ② 7 ③ or as $3 \overset{\textcircled{0}}{1} \overset{\textcircled{1}}{4} \overset{\textcircled{2}}{7} \overset{\textcircled{3}}{}$. John Napier (1550–1617) later brought the decimal point into common usage.

2-1 RATIONAL NUMBERS

When persons travel to another country, one of the first things that they learn is the monetary system. In the United States, the dollar is the basic unit, but most purchases require the use of a fractional part of a dollar. We know that a penny is \$0.01 or $\frac{1}{100}$ of a dollar, that a nickel is \$0.05 or $\frac{5}{100} = \frac{1}{20}$ of a dollar, and a dime is \$0.10 or $\frac{10}{100} = \frac{1}{10}$ of a dollar. Fractions are common in our everyday life as a part of a dollar when we make a purchase, as a part of a pound when we purchase a cut of meat, or as a part of a cup of flour when we are baking.

In our study of mathematics, we have worked with numbers that are not integers. For example, 15 minutes is $\frac{15}{60}$ or $\frac{1}{4}$ of an hour, 8 inches is $\frac{8}{12}$ or $\frac{2}{3}$ of a foot, and 8 ounces is $\frac{8}{16}$ or $\frac{1}{2}$ of a pound. These fractions are numbers in the set of rational numbers.

DEFINITION

A **rational number** is a number of the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

For every rational number $\frac{a}{b}$ that is not equal to zero, there is a **multiplicative inverse** or **reciprocal** $\frac{b}{a}$ such that $\frac{a}{b} \cdot \frac{b}{a} = 1$. Note that $\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ab}$. If the non-zero numerator of a fraction is equal to the denominator, then the fraction is equal to 1.

EXAMPLE 1

Write the multiplicative inverse of each of the following rational numbers:

Answers

a. $\frac{3}{4}$ $\frac{4}{3}$

b. $\frac{-5}{8}$ $\frac{8}{-5} = -\frac{8}{5}$

c. 5 $\frac{1}{5}$

Note that in **b**, the reciprocal of a negative number is a negative number. ◻

Decimal Values of Rational Numbers

The rational number $\frac{a}{b}$ is equivalent to $a \div b$. When a fraction or a division such as $25 \div 100$ is entered into a calculator, the decimal value is displayed. To express the quotient as a fraction, select Frac from the MATH menu. This can be done in two ways.

ENTER: 25 ÷ 100 ENTER
MATH 1 ENTER

ENTER: 25 ÷ 100 MATH 1 ENTER

DISPLAY:
```
25/100
          .25
Ans▶Frac
          1/4
```

DISPLAY:
```
25/100▶Frac
          1/4
```

When a calculator is used to evaluate a fraction such as $\frac{8}{12}$ or $8 \div 12$, the decimal value is shown as .6666666667. The calculator has rounded the value to ten decimal places, the nearest ten-billionth. The true value of $\frac{8}{12}$, or $\frac{2}{3}$, is an infinitely repeating decimal that can be written as $0.\overline{6}$. The line over the 6 means that the digit 6 repeats infinitely. Other examples of infinitely repeating decimals are:

$$0.\overline{2} = 0.22222222\ldots = \tfrac{2}{9}$$
$$0.\overline{142857} = 0.142857142857\ldots = \tfrac{1}{7}$$
$$0.\overline{12} = 0.1212121212\ldots = \tfrac{4}{33}$$
$$0.1\overline{6} = 0.1666666666\ldots = \tfrac{1}{6}$$

Every rational number is either a finite decimal or an infinitely repeating decimal. Because a finite decimal such as 0.25 can be thought of as having an infinitely repeating 0 and can be written as $0.25\overline{0}$, the following statement is true:

▶ **A number is a rational number if and only if it can be written as an infinitely repeating decimal.**

EXAMPLE 2

Find the common fractional equivalent of $0.\overline{18}$.

Solution   Let $x = 0.\overline{18} = 0.18181818\ldots$

How to Proceed

(1) Multiply the value of x by 100 to write a number in which the decimal point follows the first pair of repeating digits:

$$100x = 18.181818\ldots$$

(2) Subtract the value of x from both sides of this equation:

$$\begin{array}{rl} 100x = & 18.181818\ldots \\ -x = & -0.181818\ldots \\ \hline 99x = & 18 \end{array}$$

(3) Solve the resulting equation for x and simplify the fraction:

$$x = \tfrac{18}{99} = \tfrac{2}{11}$$

Check   The solution can be checked on a calculator.

ENTER: 2 $\div$ 11 **ENTER**

DISPLAY:

```
2/11
           .1818181818
```

Answer $\tfrac{2}{11}$

EXAMPLE 3

Express $0.12\overline{48}$ as a common fraction.

Solution: Let $x = 0.12484848\ldots$

How to Proceed

(1) Multiply the value of x by the power of 10 that makes the decimal point follow the first set of repeating digits. Since we want to move the decimal point 4 places, multiply by $10^4 = 10,000$:

$$10,000x = 1,248.4848\ldots$$

(2) Multiply the value of x by the power of 10 that makes the decimal point follow the digits that do not repeat. Since we want to move the decimal point 2 places, multiply by $10^2 = 100$:

$$100x = 12.4848\ldots$$

(3) Subtract the equation in step 2 from the equation in step 1:

$$\begin{array}{rl} 10,000x = & 1,248.4848\ldots \\ -100x = & -12.4848\ldots \\ \hline 9,900x = & 1,236 \end{array}$$

(4) Solve for x and reduce the fraction to lowest terms:

$$x = \tfrac{1,236}{9,900} = \tfrac{103}{825}$$

Answer $\tfrac{103}{825}$

> ## Procedure
>
> **To convert an infinitely repeating decimal to a common fraction:**
>
> 1. Write the equation: $x = $ decimal value.
>
> 2. Multiply both sides of the equation in step 1 by 10^m, where m is the number of places to the right of the decimal point following the first set of repeating digits.
>
> 3. Multiply both sides of the equation in step 1 by 10^n, where n is the number of places to the right of the decimal point following the *non*-repeating digits. (If there are no non-repeating digits, then let $n = 0$.)
>
> 4. Subtract the equation in step 3 from the equation in step 2.
>
> 5. Solve the resulting equation for x, and simplify the fraction completely.

Exercises

Writing About Mathematics

1. a. Why is a coin that is worth 25 cents called a quarter?

   b. Why is the number of minutes in a quarter of an hour different from the number of cents in a quarter of a dollar?

2. Explain the difference between the additive inverse and the multiplicative inverse.

Developing Skills

In 3–7, write the reciprocal (multiplicative inverse) of each given number.

3. $\frac{3}{8}$ **4.** $\frac{7}{12}$ **5.** $\frac{-2}{7}$ **6.** 8 **7.** 1

In 8–12, write each rational number as a repeating decimal.

8. $\frac{1}{6}$ **9.** $\frac{2}{9}$ **10.** $\frac{5}{7}$ **11.** $\frac{2}{15}$ **12.** $\frac{7}{8}$

In 13–22, write each decimal as a common fraction.

13. 0.125 **14.** $0.\overline{6}$ **15.** $0.\overline{2}$ **16.** $0.\overline{36}$ **17.** $0.\overline{108}$

18. $0.15\overline{6}$ **19.** $0.8\overline{3}$ **20.** $0.5\overline{7}$ **21.** $0.1\overline{36}$ **22.** $0.15\overline{90}$

2-2 SIMPLIFYING RATIONAL EXPRESSIONS

A rational number is the quotient of two integers. A **rational expression** is the quotient of two polynomials. Each of the following fractions is a rational expression:

$$\frac{3}{4} \qquad \frac{ab}{7} \qquad \frac{x+5}{2x} \qquad \frac{a^2-1}{4ab} \qquad \frac{y-2}{y^2-5y+6}$$

Division by 0 is not defined. Therefore, each of these rational expressions has no meaning when the denominator is zero. For instance:

- $\frac{x+5}{2x}$ has no meaning when $x=0$.

- $\frac{a^2-1}{4ab}$ has no meaning when $a=0$ or when $b=0$.

- $\frac{y-2}{y^2-5y+6} = \frac{y-2}{(y-2)(y-3)}$ has no meaning when $y=2$ or when $y=3$.

EXAMPLE 1

For what value or values of a is the fraction $\frac{2a-5}{3a^2-4a+1}$ undefined?

Solution A fraction is undefined or has no meaning when a factor of the denominator is equal to 0.

How to Proceed

(1) Factor the denominator: $3a^2-4a+1=(3a-1)(a-1)$

(2) Set each factor equal to 0: $3a-1=0 \qquad\qquad a-1=0$

(3) Solve each equation for a: $3a=1 \qquad\qquad a=1$

$$a=\frac{1}{3}$$

Answer The fraction is undefined when $a=\frac{1}{3}$ and when $a=1$.

▶ If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers with $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad \text{and} \qquad \frac{ac}{bd} = \frac{a}{b} \cdot \frac{c}{d} = \frac{a}{d} \cdot \frac{c}{b}$$

For example: $\frac{3}{4} \times \frac{1}{3} = \frac{3}{12}$ and $\frac{3}{12} = \frac{3 \times 1}{3 \times 4} = \frac{3}{3} \times \frac{1}{4} = 1 \times \frac{1}{4} = \frac{1}{4}.$

We can write a rational expression in simplest form by finding common factors in the numerator and denominators, as shown above.

EXAMPLE 2

Simplify: *Answers*

a. $\frac{2x}{6}$ $= \frac{2}{2} \cdot \frac{x}{3} = 1 \cdot \frac{x}{3} = \frac{x}{3}$

b. $\frac{3ab}{a(a-1)}$ $= \frac{a}{a} \cdot \frac{3b}{a-1} = 1 \cdot \frac{3b}{a-1} = \frac{3b}{a-1} \ (a \neq 0, 1)$

c. $\frac{y-2}{y^2-5y+6}$ $= \frac{y-2}{(y-2)(y-3)} = \frac{y-2}{y-2} \cdot \frac{1}{y-3} = 1 \cdot \frac{1}{y-3} = \frac{1}{y-3} \ (y \neq 2, 3)$

Note: We must eliminate any value of the variable or variables for which the denominator of the given rational expression is zero.

The rational expressions $\frac{x}{3}, \frac{3b}{a-1}$, and $\frac{1}{y-3}$ in the example shown above are in **simplest form** because there is no factor of the numerator that is also a factor of the denominator except 1 and -1. We say that the fractions have been reduced to **lowest terms**.

When the numerator or denominator of a rational expression is a monomial, each number or variable is a factor of the monomial. When the numerator or denominator of a rational expression is a polynomial with more than one term, we must factor the polynomial. Once both the numerator and denominator of the fraction are factored, we can reduce the fraction by identifying factors in the numerator that are also factors in the denominator.

In the example given above, we wrote:

$$\frac{y-2}{y^2-5y+6} = \frac{y-2}{(y-2)(y-3)} = \frac{y-2}{y-2} \cdot \frac{1}{y-3} = 1 \cdot \frac{1}{y-3} = \frac{1}{y-3} \ (y \neq 2, 3)$$

We can simplify this process by *canceling* the common factor in the numerator and denominator.

$$\frac{\overset{1}{\cancel{(y-2)}}}{\underset{1}{\cancel{(y-2)}}(y-3)} = \frac{1}{y-3} \ (y \neq 2, 3)$$

Note that canceling $(y-2)$ in the numerator and denominator of the fraction given above is the equivalent of dividing $(y-2)$ by $(y-2)$. When any number or algebraic expression that is not equal to 0 is divided by itself, the quotient is 1.

Procedure

To reduce a fraction to lowest terms:

METHOD 1

1. Factor completely both the numerator and the denominator.

2. Determine the greatest common factor of the numerator and the denominator.

3. Express the given fraction as the product of two fractions, one of which has as its numerator and its denominator the greatest common factor determined in step 2.

4. Write the fraction whose numerator and denominator are the greatest common factor as 1 and use the multiplication property of 1.

METHOD 2

1. Factor both the numerator and the denominator.

2. Divide both the numerator and the denominator by their greatest common factor by canceling the common factor.

EXAMPLE 3

Write $\frac{3x - 12}{3x}$ in lowest terms.

Solution

METHOD 1

$$\frac{3x - 12}{3x} = \frac{3(x - 4)}{3x}$$

$$= \frac{3}{3} \cdot \frac{x - 4}{x}$$

$$= 1 \cdot \frac{x - 4}{x}$$

$$= \frac{x - 4}{x}$$

METHOD 2

$$\frac{3x - 12}{3x} = \frac{\overset{1}{3}(x - 4)}{\underset{1}{3}x}$$

$$= \frac{x - 4}{x}$$

Answer $\frac{x - 4}{x}$ $(x \neq 0)$

Factors That Are Opposites

The binomials $(a - 2)$ and $(2 - a)$ are opposites or additive inverses. If we change the order of the terms in the binomial $(2 - a)$, we can write:

$$(2 - a) = (-a + 2) = -1(a - 2)$$

We can use this factored form of $(2 - a)$ to reduce the rational expression $\frac{2 - a}{a^2 - 4}$ to lowest terms.

$$\frac{2 - a}{a^2 - 4} = \frac{-1(a - 2)}{(a + 2)(a - 2)} = \frac{-1\overset{1}{(a - 2)}}{(a + 2)\underset{1}{(a - 2)}} = \frac{-1}{(a + 2)} = -\frac{1}{a + 2} \ (a \neq 2, -2)$$

EXAMPLE 4

Simplify the expression: $\dfrac{8 - 2x}{x^2 - 8x + 16}$

Solution

METHOD 1

$$\frac{8 - 2x}{x^2 - 8x + 16} = \frac{2(4 - x)}{(x - 4)(x - 4)}$$

$$= \frac{2(-1)(x - 4)}{(x - 4)(x - 4)}$$

$$= \frac{(x - 4)}{(x - 4)} \cdot \frac{-2}{(x - 4)}$$

$$= \frac{-2}{x - 4}$$

METHOD 2

$$\frac{8 - 2x}{x^2 - 8x + 16} = \frac{2(4 - x)}{(x - 4)(x - 4)}$$

$$= \frac{2(-1)(x - 4)}{(x - 4)(x - 4)}$$

$$= \frac{-2\overset{1}{(x - 4)}}{(x - 4)\underset{1}{(x - 4)}}$$

$$= \frac{-2}{x - 4}$$

Answer $\dfrac{-2}{x - 4} \ (x \neq 4)$

Exercises

Writing About Mathematics

1. Abby said that $\frac{3x}{3x + 4}$ can be reduced to lowest terms by canceling $3x$ so that the result is $\frac{1}{4}$. Do you agree with Abby? Explain why or why not.

2. Does $\frac{2a - 3}{2a - 3} = 1$ for all real values of a? Justify your answer.

Developing Skills

In 3–10, list the values of the variables for which the rational expression is undefined.

3. $\dfrac{5a^2}{3a}$

4. $\dfrac{-2d}{6c}$

5. $\dfrac{a + 2}{ab}$

6. $\dfrac{x - 5}{x + 5}$

7. $\dfrac{2a}{2a - 7}$

8. $\dfrac{b + 3}{b^2 + b - 6}$

9. $\dfrac{4c}{2c^2 - 2c}$

10. $\dfrac{5}{x^3 - 5x^2 - 6x}$

In 11–30, write each rational expression in simplest form and list the values of the variables for which the fraction is undefined.

11. $\frac{6}{10}$

12. $\frac{5a^2b}{10a}$

13. $\frac{12xy^2}{3x^2y}$

14. $\frac{14b^4}{21b^3}$

15. $\frac{9cd^2}{12c^4d^2}$

16. $\frac{8a + 16}{12a}$

17. $\frac{9y^2 + 3y}{6y^2}$

18. $\frac{8ab - 4b^2}{6ab}$

19. $\frac{10d}{15d - 20d^2}$

20. $\frac{8c^2}{8c^2 + 16c}$

21. $\frac{3xy}{9xy + 6x^2y^3}$

22. $\frac{2a + 10}{3a + 15}$

23. $\frac{4a^2 - 16}{4a + 8}$

24. $\frac{x^2 - 7x + 12}{x^2 + 2x - 15}$

25. $\frac{5y^2 - 20}{y^2 + 4y + 4}$

26. $\frac{-7 + 7a}{21a^2 - 21}$

27. $\frac{a^3 - a^2 - a + 1}{a^2 - 2a + 1}$

28. $\frac{3 - (b + 1)}{4 - b^2}$

29. $\frac{4 - 2(x - 1)}{x^2 - 6x + 9}$

30. $\frac{5(1 - b) + 15}{b^2 - 16}$

2-3 MULTIPLYING AND DIVIDING RATIONAL EXPRESSIONS

Multiplying Rational Expressions

We know that $\frac{2}{3} \times \frac{4}{5} = \frac{8}{15}$ and that $\frac{3}{4} \times \frac{3}{2} = \frac{9}{8}$. In general, the product of two rational numbers $\frac{a}{b}$ and $\frac{c}{d}$ is $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$ for $b \neq 0$ and $d \neq 0$.

This same rule holds for the product of two rational expressions:

▶ **The product of two rational expressions is a fraction whose numerator is the product of the given numerators and whose denominator is the product of the given denominators.**

For example:

$$\frac{3(a + 5)}{4a} \cdot \frac{12}{a} = \frac{36(a + 5)}{4a^2} \quad (a \neq 0)$$

This product can be reduced to lowest terms by dividing numerator and denominator by the common factor, 4.

$$\frac{3(a + 5)}{4a} \cdot \frac{12}{a} = \frac{36(a + 5)}{4a^2} = \frac{\overset{9}{36}(a + 5)}{\underset{1}{4}a^2} = \frac{9(a + 5)}{a^2}$$

We could have canceled the factor 4 before we multiplied, as shown below.

$$\frac{3(a + 5)}{4a} \times \frac{12}{a} = \frac{3(a + 5)}{\underset{1}{4}a} \times \frac{\overset{3}{12}}{a} = \frac{9(a + 5)}{a^2}$$

Note that a is not a common factor of the numerator and denominator because it is *one term* of the factor $(a + 5)$, not a factor of the numerator.

> ### Procedure
>
> **To multiply fractions:**
>
> **METHOD 1**
>
> 1. Multiply the numerators of the given fractions and the denominators of the given fractions.
>
> 2. Reduce the resulting fraction, if possible, to lowest terms.
>
> **METHOD 2**
>
> 1. Factor any polynomial that is not a monomial.
>
> 2. Cancel any factors that are common to a numerator and a denominator.
>
> 3. Multiply the resulting numerators and the resulting denominators to write the product in lowest terms.

EXAMPLE 1

a. Find the product of $\dfrac{12b^2}{5b + 15} \cdot \dfrac{b^2 - 9}{3b^2 + 9b}$ in simplest form.

b. For what values of the variable are the given fractions and the product undefined?

Solution **a. METHOD 1**

How to Proceed

(1) Multiply the numerators of the fractions and the denominators of the fractions:

$$\frac{12b^2}{5b + 15} \cdot \frac{b^2 - 9}{3b^2 + 9b} = \frac{12b^2(b^2 - 9)}{(5b + 15)(3b^2 + 9b)}$$

(2) Factor the numerator and the denominator. Note that the factors of $(5b + 15)$ are $5(b + 3)$ and the factors of $(3b^2 + 9b)$ are $3b(b + 3)$. Reduce the resulting fraction to lowest terms:

$$= \frac{12b^2(b + 3)(b - 3)}{15b(b + 3)(b + 3)}$$

$$= \frac{3b(b + 3)}{3b(b + 3)} \cdot \frac{4b(b - 3)}{5(b + 3)}$$

$$= 1 \cdot \frac{4b(b - 3)}{5(b + 3)}$$

$$= \frac{4b(b - 3)}{5(b + 3)} \quad \textit{Answer}$$

METHOD 2

How to Proceed

(1) Factor each binomial term completely:

$$\frac{12b^2}{5b + 15} \cdot \frac{b^2 - 9}{3b^2 + 9b} = \frac{12b^2}{5(b + 3)} \cdot \frac{(b + 3)(b - 3)}{3b(b + 3)}$$

(2) Cancel any factors that are common to a numerator and a denominator:

$$= \frac{\overset{4}{\cancel{12}}\overset{b}{b^2}}{5(b + 3)} \cdot \frac{\overset{1}{\cancel{(b + 3)}}(b - 3)}{\underset{1\ 1}{3\cancel{b}}\underset{1}{(\cancel{b + 3})}}$$

(3) Multiply the remaining factors:

$$= \frac{4b(b - 3)}{5(b + 3)} \quad \textit{Answer}$$

b. The given fractions and their product are undefined when $b = 0$ and when $b = -3$. *Answer*

Dividing Rational Expressions

We can divide two rational numbers or two rational expressions by changing the division to a related multiplication. Let $\frac{a}{b}$ and $\frac{c}{d}$ be two rational numbers or rational expressions with $b \neq 0, c \neq 0, d \neq 0$. Since $\frac{c}{d}$ is a non-zero rational number or rational expression, there exists a multiplicative inverse $\frac{d}{c}$ such that $\frac{c}{d} \cdot \frac{d}{c} = 1$.

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}} = \frac{\frac{a}{b} \cdot \frac{d}{c}}{\frac{c}{d} \cdot \frac{d}{c}} = \frac{\frac{a \cdot d}{b \cdot c}}{\frac{c \cdot d}{d \cdot c}} = \frac{\frac{a \cdot d}{b \cdot c}}{1} = \frac{a}{b} \cdot \frac{d}{c}$$

We have just derived the following procedure.

> **Procedure**
>
> **To divide two rational numbers or rational expressions, multiply the dividend by the reciprocal of the divisor.**

For example:

$$\frac{x + 5}{2x^2} \div \frac{5}{x + 1} = \frac{x + 5}{2x^2} \cdot \frac{x + 1}{5}$$

$$= \frac{(x + 5)(x + 1)}{10x^2} \quad (x \neq -1, 0)$$

This product is in simplest form because the numerator and denominator have no common factors.

Recall that a can be written as $\frac{a}{1}$ and therefore, if $a \neq 0$, the reciprocal of a is $\frac{1}{a}$. For example:

$$\frac{2b - 2}{5} \div (b - 1) = \frac{2(b - 1)}{5} \cdot \frac{1}{b - 1} = \frac{2\overset{1}{\cancel{(b - 1)}}}{5} \cdot \frac{1}{\underset{1}{\cancel{b - 1}}} = \frac{2}{5} \; (b \neq 1)$$

EXAMPLE 2

Divide and simplify: $\dfrac{a^2 - 3a - 10}{5a} \div \dfrac{a^2 - 5a}{15}$

Solution   *How to Proceed*

(1) Use the reciprocal of the divisor to write the division as a multiplication:

$$\dfrac{a^2 - 3a - 10}{5a} \div \dfrac{a^2 - 5a}{15} = \dfrac{a^2 - 3a - 10}{5a} \cdot \dfrac{15}{a^2 - 5a}$$

(2) Factor each polynomial:

$$= \dfrac{(a - 5)(a + 2)}{5a} \cdot \dfrac{15}{a(a - 5)}$$

(3) Cancel any factors that are common to the numerator and denominator:

$$= \dfrac{\overset{1}{(a - 5)}(a + 2)}{5a} \cdot \dfrac{\overset{3}{15}}{a\overset{}{(a - 5)}}_{\,1}$$

(4) Multiply the remaining factors:

$$= \dfrac{3(a + 2)}{a^2}$$

Answer $\dfrac{3(a + 2)}{a^2}$ $(a \neq 0, 5)$

EXAMPLE 3

Perform the indicated operations and write the answer in simplest form:

$$\dfrac{3a}{a + 3} \div \dfrac{5a}{2a + 6} \cdot \dfrac{3}{a}$$

Solution Recall that multiplications and divisions are performed in the order in which they occur from left to right.

$$\dfrac{3a}{a + 3} \div \dfrac{5a}{2a + 6} \cdot \dfrac{3}{a} = \dfrac{3a}{a + 3} \cdot \dfrac{2(a + 3)}{5a} \cdot \dfrac{a}{3}$$

$$= \dfrac{3a}{a + 3} \cdot \dfrac{2(a + 3)}{5a} \cdot \dfrac{a}{3}$$

$$= \dfrac{6}{5} \cdot \dfrac{a}{3}$$

$$= \dfrac{2a}{5} \quad (a \neq -3, 0) \quad \textit{Answer}$$

Exercises

Writing About Mathematics

1. Joshua wanted to write this division in simplest form: $\frac{3}{x-2} \div \frac{4(x-2)}{7}$. He began by canceling $(x-2)$ in the numerator and denominator and wrote following:

$$\frac{3}{\overset{}{\underset{1}{x-2}}} \div \frac{4(\overset{1}{x-2})}{7} = \frac{3}{1} \div \frac{4}{7} = \frac{3}{1} \times \frac{7}{4} = \frac{21}{4}$$

   Is Joshua's answer correct? Justify your answer.

2. Gabriel wrote $\frac{12x}{5x+10} \div \frac{4}{5} = \frac{12x \div 4}{(5x+10) \div 5} = \frac{3x}{x+2}$. Is Gabriel's solution correct? Justify your answer.

Developing Skills

In 3–12, multiply and express each product in simplest form. In each case, list any values of the variables for which the fractions are not defined.

3. $\frac{2}{3} \times \frac{3}{4}$

4. $\frac{5}{7a} \cdot \frac{3a}{20}$

5. $\frac{4y}{5x} \cdot \frac{x}{8y}$

6. $\frac{3a}{5} \cdot \frac{10}{9a}$

7. $\frac{b+1}{4} \cdot \frac{12}{5b+5}$

8. $\frac{a^2-100}{3a} \cdot \frac{a^2}{2a-20}$

9. $\frac{7y+21}{7y} \cdot \frac{3}{y^2-9}$

10. $\frac{a^2-5a+4}{3a+6} \cdot \frac{2a+4}{a^2-16}$

11. $\frac{2a+4}{6a} \cdot \frac{3a^2}{a^2+2a}$

12. $\frac{6-2x}{x^2-9} \cdot \frac{15+5x}{4x}$

In 13–24, divide and express each quotient in simplest form. In each case, list any values of the variables for which the fractions are not defined.

13. $\frac{3}{4} \div \frac{9}{20}$

14. $\frac{12}{a} \div \frac{6}{4a}$

15. $\frac{6b}{5c} \div \frac{3b}{10c}$

16. $\frac{a^2}{8a} \div \frac{3a}{4}$

17. $\frac{x-2}{3x} \div \frac{4x-8}{9}$

18. $\frac{6y^2-3y}{3y} \div \frac{4y^2-1}{2}$

19. $\frac{c^2-6c+9}{5c-15} \div \frac{c-3}{5}$

20. $\frac{w^2-w}{5w} \div \frac{w^2-1}{5}$

21. $\frac{4b+12}{b} \div (b+3)$

22. $\frac{a^2+8a+15}{4a} \div (a+3)$

23. $(2x+7) \div \frac{1}{2x^2+5x-7}$

24. $(a^2-1) \div \frac{2a+2}{a}$

In 25–30, perform the indicated operations and write the result in simplest form. In each case, list any values of the variables for which the fractions are not defined.

25. $\frac{3}{5} \times \frac{5}{9} \div \frac{4}{3}$

26. $\frac{3x}{x-1} \cdot \frac{x^2-1}{x} \div \frac{x+1}{3}$

27. $\frac{2a}{a+2} \cdot \frac{a^2-4}{4a^2} \div \frac{a-2}{a}$

28. $(x^2 - 2x + 1) \div \frac{x-1}{3} \cdot \frac{x+4}{3x}$

29. $(3b)^2 \div \frac{3b}{b+2} \cdot \frac{2b+4}{b}$

30. $\frac{x^2-3x+2}{4x} \cdot \frac{12x^2}{x^2-2x} \div \frac{x-1}{x}$

2-4 ADDING AND SUBTRACTING RATIONAL EXPRESSIONS

We know that $\frac{3}{7} = 3\left(\frac{1}{7}\right)$ and that $\frac{2}{7} = 2\left(\frac{1}{7}\right)$. Therefore:

$$\frac{3}{7} + \frac{2}{7} = 3\left(\frac{1}{7}\right) + 2\left(\frac{1}{7}\right)$$
$$= (3+2)\left(\frac{1}{7}\right)$$
$$= 5\left(\frac{1}{7}\right)$$
$$= \frac{5}{7}$$

We can also write:

$$\frac{x+2}{x} + \frac{2x-5}{x} = (x+2)\left(\frac{1}{x}\right) + (2x-5)\left(\frac{1}{x}\right)$$
$$= (x+2+2x-5)\left(\frac{1}{x}\right)$$
$$= \frac{3x-3}{x} \quad (x \neq 0)$$

In general, the sum of any two fractions with a common denominator is a fraction whose numerator is the sum of the numerators and whose denominator is the common denominator.

$$\frac{3a}{a+1} + \frac{a}{a+1} = \frac{4a}{a+1} \quad (a \neq -1)$$

In order to add two fractions that do *not* have a common denominator, we need to change one or both of the fractions to equivalent fractions that *do* have a common denominator. That common denominator can be the product of the given denominators.

For example, to add $\frac{1}{5} + \frac{1}{7}$, we change each fraction to an equivalent fraction whose denominator is 35 by multiplying each fraction by a fraction equal to 1.

$$\frac{1}{5} + \frac{1}{7} = \frac{1}{5} \times \frac{7}{7} + \frac{1}{7} \times \frac{5}{5}$$
$$= \frac{7}{35} + \frac{5}{35}$$
$$= \frac{12}{35}$$

To add the fractions $\frac{5}{2x}$ and $\frac{3}{y}$, we need to find a denominator that is a multiple of both $2x$ and of y. One possibility is their product, $2xy$. Multiply each fraction by a fraction equal to 1 so that the denominator of each fraction will be $2xy$:

$$\frac{5}{2x} = \frac{5}{2x} \cdot \frac{y}{y} = \frac{5y}{2xy} \quad \text{and} \quad \frac{3}{y} = \frac{3}{y} \cdot \frac{2x}{2x} = \frac{6x}{2xy}$$

$$\frac{5}{2x} + \frac{3}{y} = \frac{5y}{2xy} + \frac{6x}{2xy} = \frac{5y + 6x}{2xy} \quad (x \neq 0, y \neq 0)$$

The **least common denominator** (**LCD**) is often smaller than the product of the two denominators. It is the **least common multiple** (**LCM**) of the denominators, that is, the product of all factors of one or both of the denominators.

For example, to add $\frac{1}{2a + 2} + \frac{1}{a^2 - 1}$, first find the factors of each denominator. The least common denominator is the product of all of the factors of the first denominator times all factors of the second that are not factors of the first. Then multiply each fraction by a fraction equal to 1 so that the denominator of each fraction will be equal to the LCD.

$$\begin{aligned}
\text{Factors of } 2a + 2: \quad & 2 \cdot (a + 1) \\
\text{Factors of } a^2 - 1: \quad & (a + 1) \cdot (a - 1) \\
\text{LCD:} \quad & 2 \cdot (a + 1) \cdot (a - 1)
\end{aligned}$$

$$\frac{1}{2a + 2} = \frac{1}{2(a + 1)} \cdot \frac{a - 1}{a - 1} \quad \text{and} \quad \frac{1}{a^2 - 1} = \frac{1}{(a + 1)(a - 1)} \cdot \frac{2}{2}$$

$$= \frac{a - 1}{2(a + 1)(a - 1)} \qquad\qquad\qquad = \frac{2}{2(a + 1)(a - 1)}$$

$$\frac{1}{2a + 2} + \frac{1}{a^2 - 1} = \frac{a - 1 + 2}{2(a + 1)(a - 1)} = \frac{a + 1}{2(a + 1)(a - 1)}$$

Since this sum has a common factor in the numerator and denominator, it can be reduced to lowest terms.

$$\frac{a + 1}{2(a + 1)(a - 1)} = \frac{\overset{1}{\cancel{a + 1}}}{2\underset{1}{(\cancel{a + 1})}(a - 1)} = \frac{1}{2(a - 1)} \quad (a \neq -1, a \neq 1)$$

Any polynomial can be written as a rational expression with a denominator of 1. To add a polynomial to a rational expression, write the polynomial as an equivalent rational expression.

For example, to write the sum $b + 3 + \frac{1}{2b}$ as a single fraction, multiply $(b + 3)$ by 1 in the form $\frac{2b}{2b}$.

$$\begin{aligned}
(b + 3) + \frac{1}{2b} &= \frac{b + 3}{1}\left(\frac{2b}{2b}\right) + \frac{1}{2b} \\
&= \frac{2b^2 + 6b}{2b} + \frac{1}{2b} \\
&= \frac{2b^2 + 6b + 1}{2b} \quad (b \neq 0)
\end{aligned}$$

EXAMPLE 1

Write the difference $\dfrac{x}{x^2 - 4x + 3} - \dfrac{x}{x^2 + 2x - 3}$ as a single fraction in lowest terms.

Solution

How to Proceed

(1) Find the LCD of the fractions:

$$x^2 - 4x + 3 = (x - 3) \cdot (x - 1)$$
$$x^2 + 2x - 3 = (x - 1) \cdot (x + 3)$$
$$\text{LCD} = (x - 3) \cdot (x - 1) \cdot (x + 3)$$

(2) Write each fraction as an equivalent fraction with a denominator equal to the LCD:

$$\dfrac{x}{x^2 - 4x + 3} = \dfrac{x}{(x - 3)(x - 1)} \cdot \dfrac{x + 3}{x + 3}$$
$$= \dfrac{x^2 + 3x}{(x - 3)(x - 1)(x + 3)}$$

$$\dfrac{x - 3}{x^2 + 2x - 3} = \dfrac{x - 3}{(x + 3)(x - 1)} \cdot \dfrac{x - 3}{x - 3}$$
$$= \dfrac{x^2 - 6x + 9}{(x - 3)(x - 1)(x + 3)}$$

(3) Subtract:

$$\dfrac{x}{x^2 - 4x + 3} - \dfrac{x - 3}{x^2 + 2x - 3}$$
$$= \dfrac{x^2 + 3x - (x^2 - 6x + 9)}{(x - 3)(x - 1)(x + 3)}$$

(4) Simplify:

$$= \dfrac{9x - 9}{(x - 3)(x - 1)(x + 3)}$$

(5) Reduce to lowest terms:

$$= \dfrac{9}{(x - 3)(x + 3)}$$

Answer $\dfrac{9}{(x - 3)(x + 3)}$ $(x \neq -3, 1, 3)$

EXAMPLE 2

Simplify: $\left(x - \dfrac{1}{x}\right)\left(1 + \dfrac{1}{x - 1}\right)$

Solution **STEP 1.** Rewrite each expression in parentheses as a single fraction.

$$x - \dfrac{1}{x} = x\left(\dfrac{x}{x}\right) - \dfrac{1}{x} \quad \text{and} \quad 1 + \dfrac{1}{x - 1} = 1\left(\dfrac{x - 1}{x - 1}\right) + \dfrac{1}{x - 1}$$
$$= \dfrac{x^2}{x} - \dfrac{1}{x} \qquad\qquad\qquad\qquad = \dfrac{x - 1}{x - 1} + \dfrac{1}{x - 1}$$
$$= \dfrac{x^2 - 1}{x} \qquad\qquad\qquad\qquad\quad = \dfrac{x - 1 + 1}{x - 1}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad\quad = \dfrac{x}{x - 1}$$

STEP 2. Multiply.

$$\left(\dfrac{x^2 - 1}{x}\right)\left(\dfrac{x}{x - 1}\right) = \left(\dfrac{(x + 1)(x - 1)}{x}\right)\left(\dfrac{x}{x - 1}\right)$$
$$= \left(\dfrac{(x + 1)\overset{1}{\cancel{(x - 1)}}}{\cancel{x}}\right)\left(\dfrac{\overset{1}{\cancel{x}}}{\cancel{x - 1}}\right)$$
$$= \dfrac{x + 1}{1}$$
$$= x + 1$$

Alternative **STEP 1.** Multiply using the distributive property.
Solution

$$\left(x - \tfrac{1}{x}\right)\left(1 + \tfrac{1}{x-1}\right) = x\left(1 + \tfrac{1}{x-1}\right) - \tfrac{1}{x}\left(1 + \tfrac{1}{x-1}\right)$$

$$= x + \frac{x}{x-1} - \frac{1}{x} - \frac{1}{x(x-1)}$$

STEP 2. Add the fractions. The least common denominator is $x(x-1)$.

$$x + \frac{x}{x-1} - \frac{1}{x} - \frac{1}{x(x-1)} = x\left(\frac{x(x-1)}{x(x-1)}\right) + \frac{x}{(x-1)}\left(\frac{x}{x}\right) - \frac{1}{x}\left(\frac{x-1}{x-1}\right) - \frac{1}{x(x-1)}$$

$$= \frac{x^3 - x^2}{x(x-1)} + \frac{x^2}{x(x-1)} - \frac{x-1}{x(x-1)} - \frac{1}{x(x-1)}$$

$$= \frac{x^3 - x^2 + x^2 - x + 1 - 1}{x(x-1)}$$

$$= \frac{x^3 - x}{x(x-1)}$$

STEP 3. Simplify.

$$\frac{x^3 - x}{x(x-1)} = \frac{\overset{1}{\cancel{x}}(\cancel{x-1})(x+1)}{\underset{1}{\cancel{x}}(\underset{1}{\cancel{x-1}})} = x + 1$$

Answer $x + 1$ $(x \neq 0, 1)$ ◻

Exercises

Writing About Mathematics

1. Ashley said that $\frac{(a+2)(a-1)}{(a+3)(a-1)} = \frac{a+2}{a+3}$ for all values of a except $a = -3$. Do you agree with Ashley? Explain why or why not.

2. Matthew said that $\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$ when $b \neq 0, d \neq 0$. Do you agree with Matthew? Justify your answer.

Developing Skills

In 3–20, perform the indicated additions or subtractions and write the result in simplest form. In each case, list any values of the variables for which the fractions are not defined.

3. $\frac{x}{3} + \frac{2x}{3}$

4. $\frac{2x^2 + 1}{5x} - \frac{7x^2 - 1}{5x}$

5. $\frac{x}{7} + \frac{x}{3}$

6. $\frac{a-1}{5} - \frac{a+1}{4}$

7. $\frac{y+2}{2} + \frac{2y-3}{3}$

8. $\frac{a+5}{5a} - \frac{a-8}{8a}$

9. $\frac{2x+3}{6x} - \frac{x-2}{4x}$

10. $\frac{a+1}{3a} + \frac{3}{2}$

11. $3 + \frac{2}{x}$

12. $5 - \frac{1}{2y}$

13. $a - \frac{3}{2a}$

14. $\frac{1}{x} + 1$

15. $\frac{3}{x+2} + \frac{x-2}{x}$

16. $\frac{b}{b-1} - \frac{1}{2-2b}$

17. $\frac{1}{2-x} + \frac{2}{x-2}$

18. $\frac{1}{a^2-a-6} - \frac{1}{2a^2-7a+3}$

19. $\frac{2}{a^2-4} - \frac{1}{a^2+2a}$

20. $\frac{1}{x} + \frac{1}{x-2} - \frac{2}{x^2-2x}$

Applying Skills

In 21–24, the length and width of a rectangle are expressed in terms of a variable.

a. Express each perimeter in terms of the variable.

b. Express each area in terms of the variable.

21. $l = 2x$ and $w = \frac{1}{x}$

22. $l = 3x + 3$ and $w = \frac{1}{3}$

23. $l = \frac{x}{x - 1}$ and $w = \frac{3}{x - 1}$

24. $l = \frac{x}{x + 1}$ and $w = \frac{x}{x + 2}$

2-5 RATIO AND PROPORTION

We often want to compare two quantities that use the same unit. For example, in a given class of 25 students, there are 11 students who are boys. We can say that $\frac{11}{25}$ of the students are boys or that the ratio of students who are boys to all students in the class is 11 : 25.

DEFINITION

A **ratio** is the comparison of two numbers by division. The ratio of a to b can be written as $\frac{a}{b}$ or as $a : b$ when $b \neq 0$.

A ratio, like a fraction, can be simplified by dividing each term by the same non-zero number. A ratio is in **simplest form** when the terms of the ratio are integers that have no common factor other than 1.

For example, to write the ratio of 3 inches to 1 foot, we must first write each measure in terms of the same unit and then divide each term of the ratio by a common factor.

$$\frac{3 \text{ inches}}{1 \text{ foot}} = \frac{3 \text{ inches}}{1 \text{ foot}} \times \frac{1 \text{ foot}}{12 \text{ inches}} = \frac{3}{12} = \frac{1}{4}$$

In lowest terms, the ratio of 3 inches to 1 foot is 1 : 4.

An equivalent ratio can also be written by multiplying each term of the ratio by the same non-zero number. For example, $4 : 7 = 4(2) : 7(2) = 8 : 14$.

In general, for $x \neq 0$:

$$a : b = ax : bx$$

EXAMPLE I

The length of a rectangle is 1 yard and the width is 2 feet. What is the ratio of length to width of this rectangle?

Solution   The ratio must be in terms of the same measure.

$$\frac{1 \text{ yd}}{2 \text{ ft}} \times \frac{3 \text{ ft}}{1 \text{ yd}} = \frac{3}{2}$$

Answer   The ratio of length to width is $3 : 2$.

EXAMPLE 2

The ratio of the length of one of the congruent sides of an isosceles triangle to the length of the base is $5 : 2$. If the perimeter of the triangle is 42.0 centimeters, what is the length of each side?

Solution   Let AB and BC be the lengths of the congruent sides of isosceles $\triangle ABC$ and AC be the length of the base.

$AB : AC = 5 : 2 = 5x : 2x$

Therefore, $AB = 5x$,

$BC = 5x$,

and $AC = 2x$.

$$
\begin{array}{c|cc}
AB + BC + AC = \text{Perimeter} & AB = BC = 5(3.5) & AC = 2(3.5) \\
5x + 5x + 2x = 42 & = 17.5 \text{ cm} & = 7.0 \text{ cm} \\
12x = 42 & & \\
x = 3.5 \text{ cm} & &
\end{array}
$$
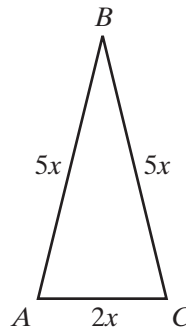
Check $AB + BC + AC = 17.5 + 17.5 + 7.0 = 42.0 \text{ cm} ✔$

Answer   The sides measure 17.5, 17.5, and 7.0 centimeters.

Proportion

An equation that states that two ratios are equal is called a **proportion**. For example, $3 : 12 = 1 : 4$ is a proportion. This proportion can also be written as $\frac{3}{12} = \frac{1}{4}$.

In general, if $b \neq 0$ and $d \neq 0$, then $a : b = c : d$ or $\frac{a}{b} = \frac{c}{d}$ are proportions. The first and last terms, a and d, are called the **extremes** of the proportion and the second and third terms, b and c, are the **means** of the proportion.

If we multiply both sides of the proportion by the product of the second and last terms, we can prove a useful relationship among the terms of a proportion.

$$\frac{a}{b} = \frac{c}{d}$$

$$bd\left(\frac{a}{b}\right) = bd\left(\frac{c}{d}\right)$$

$$\overset{1}{b}d\left(\frac{a}{\underset{1}{b}}\right) = b\overset{1}{d}\left(\frac{c}{\underset{1}{d}}\right)$$

$$da = bc$$

▶ **In any proportion, the product of the means is equal to the product of the extremes.**

EXAMPLE 3

In the junior class, there are 24 more girls than boys. The ratio of girls to boys is $5 : 4$. How many girls and how many boys are there in the junior class?

Solution

How to Proceed

(1) Use the fact that the number of girls is 24 more than the number of boys to represent the number of girls and of boys in terms of x:

Let $x =$ the number of boys,

$x + 24 =$ the number of girls.

(2) Write a proportion. Set the ratio of the number of boys to the number of girls, in terms of x, equal to the given ratio:

$$\frac{x + 24}{x} = \frac{5}{4}$$

(3) Use the fact that the product of the means is equal to the product of the extremes. Solve the equation:

$$5x = 4(x + 24)$$
$$5x = 4x + 96$$
$$x = 96$$
$$x + 24 = 96 + 24$$
$$= 120$$

Alternative (1) Use the given ratio to represent the   Let $5x$ = the number of girls
Solution   number of boys and the number of girls
            in terms of x: $4x$ = the number of boys

(2) Use the fact that the number of girls $5x = 24 + 4x$
    is 24 more than the number of boys to
    write an equation. Solve the equation $x = 24$
    for x:

(3) Use the value of x to find the number $5x = 5(24) = 120$ girls
    of girls and the number of boys: $4x = 4(24) = 96$ boys

Answer   There are 120 girls and 96 boys in the junior class.

Exercises

Writing About Mathematics

1. If $\frac{a}{b} = \frac{c}{d}$, then is $\frac{a}{c} = \frac{b}{d}$? Justify your answer.

2. If $\frac{a}{b} = \frac{c}{d}$, then is $\frac{a+b}{b} = \frac{c+d}{d}$? Justify your answer.

Developing Skills

In 3–10, write each ratio in simplest form.

3. $12 : 8$

4. $21 : 14$

5. $3 : 18$

6. $15 : 75$

7. $6a : 9a, a \neq 0$

8. $\frac{18}{27}$

9. $\frac{24}{72}$

10. $\frac{10x}{35x}, x \neq 0$

In 11–19, solve each proportion for the variable.

11. $\frac{x}{8} = \frac{6}{24}$

12. $\frac{x-1}{15} = \frac{2}{5}$

13. $\frac{a-2}{2a} = \frac{5}{14}$

14. $\frac{y+3}{y+8} = \frac{6}{15}$

15. $\frac{3x+3}{16} = \frac{2x+1}{10}$

16. $\frac{2}{x-1} = \frac{x+2}{2}$

17. $\frac{4x-8}{3} = \frac{8}{x-3}$

18. $\frac{x-2}{x} = \frac{3}{x+2}$

19. $\frac{x}{5} = \frac{x+4}{x+13}$

Applying Skills

20. The ratio of the length to the width of a rectangle is $5 : 4$. The perimeter of the rectangle is 72 inches. What are the dimensions of the rectangle?

21. The ratio of the length to the width of a rectangle is $7 : 3$. The area of the rectangle is 336 square centimeters. What are the dimensions of the rectangle?

22. The basketball team has played 21 games. The ratio of wins to losses is $5 : 2$. How many games has the team won?

23. In the chess club, the ratio of boys to girls is 6 : 5. There are 3 more boys than girls in the club. How many members are in the club?

24. Every year, Javier makes a total contribution of $125 to two local charities. The two donations are in the ratio of 3 : 2. What contribution does Javier make to each charity?

25. A cookie recipe uses flour and sugar in the ratio of 9 : 4. If Nicholas uses 1 cup of sugar, how much flour should he use?

26. The directions on a bottle of cleaning solution suggest that the solution be diluted with water. The ratio of solution to water is 1 : 7. How many cups of solution and how many cups of water should Christopher use to make 2 gallons (32 cups) of the mixture?

2-6 COMPLEX RATIONAL EXPRESSIONS

A **complex fraction** is a fraction whose numerator, denominator, or both contain fractions. Some examples of complex fractions are:

$$\frac{\frac{1}{3}}{2} \qquad \frac{5}{\frac{2}{7}} \qquad \frac{2\frac{3}{4}}{\frac{1}{8}}$$

A complex fraction can be simplified by multiplying by a fraction equal to 1; that is, by a fraction whose non-zero numerator and denominator are equal. The numerator and denominator of this fraction should be a common multiple of the denominators of the fractional terms. For example:

$$\frac{\frac{1}{3}}{2} = \frac{\frac{1}{3}}{2} \times \frac{3}{3} = \frac{1}{6} \qquad \frac{5}{\frac{2}{7}} = \frac{5}{\frac{2}{7}} \times \frac{7}{7} = \frac{35}{2} \qquad \frac{2\frac{3}{4}}{\frac{1}{8}} = \frac{2 + \frac{3}{4}}{\frac{1}{8}} \times \frac{8}{8} = \frac{16 + 6}{1} = 22$$

A complex fraction can also be simplified by dividing the numerator by the denominator.

$$\frac{\frac{1}{3}}{2} = \frac{1}{3} \div \frac{2}{1} = \frac{1}{3} \times \frac{1}{2} = \frac{1}{6} \qquad \frac{5}{\frac{2}{7}} = \frac{5}{1} \div \frac{2}{7} = 5 \times \frac{7}{2} = \frac{35}{2}$$

$$\frac{2\frac{3}{4}}{\frac{1}{8}} = \frac{\frac{8}{4} + \frac{3}{4}}{\frac{1}{8}} = \frac{\frac{11}{4}}{\frac{1}{8}} = \frac{11}{4} \div \frac{1}{8} = \frac{11}{4} \times \frac{8}{1} = \frac{88}{4} = 22$$

A **complex rational expression** has a rational expression in the numerator, the denominator, or both. For example, the following are complex rational expressions.

$$\frac{\frac{1}{a}}{a} \qquad \frac{x + 3}{\frac{1}{x}} \qquad \frac{1 + \frac{1}{b} - \frac{2}{b^2}}{\frac{1}{b}}$$

Like complex fractions, complex rational expressions can be simplified by multiplying numerator and denominator by a common multiple of the denominators in the fractional terms.

$$\frac{\frac{1}{a}}{a} = \frac{1}{a} \cdot \frac{a}{a}$$

$$= \frac{1}{a^2}$$

$$(a \neq 0)$$

$$\frac{x+3}{\frac{1}{x}} = \frac{x+3}{\frac{1}{x}} \cdot \frac{x}{x}$$

$$= \frac{x^2 + 3x}{1}$$

$$= x^2 + 3x$$

$$(x \neq 0)$$

$$\frac{1 + \frac{1}{b} - \frac{2}{b^2}}{1 - \frac{1}{b}} = \frac{1 + \frac{1}{b} - \frac{2}{b^2}}{1 - \frac{1}{b}} \cdot \frac{b^2}{b^2}$$

$$= \frac{b^2 + b - 2}{b^2 - b}$$

$$= \frac{(b+2)(b-1)}{b(b-1)}$$

$$= \frac{(b+2)\overset{1}{\cancel{(b-1)}}}{b\underset{1}{\cancel{(b-1)}}}$$

$$= \frac{b+2}{b} \quad (b \neq 0, 1)$$

Alternatively, a complex rational expression can also be simplified by dividing the numerator by the denominator. Choose the method that is easier for each given expression.

Procedure

To simplify a complex fraction:
METHOD 1
Multiply by $\frac{m}{m}$, where m is the least common multiple of the denominators of the fractional terms.
METHOD 2
Multiply the numerator by the reciprocal of the denominator of the fraction.

EXAMPLE I

Express $\dfrac{\frac{2}{a}}{\frac{3}{4a^2}}$ in simplest form.

Solution **METHOD 1**

Multiply numerator and denominator of the fraction by the least common multiple of the denominators of the fractional terms. The least common multiple of a and $4a^2$ is $4a^2$.

$$\frac{\frac{2}{a}}{\frac{3}{4a^2}} = \frac{\frac{2}{a}}{\frac{3}{4a^2}} \cdot \frac{4a^2}{4a^2} = \frac{8a}{3}$$

METHOD 2

Write the fraction as the numerator divided by the denominator. Change the division to multiplication by the reciprocal of the divisor.

$$\dfrac{\frac{2}{a}}{\frac{3}{4a^2}} = \dfrac{2}{a} \div \dfrac{3}{4a^2} = \dfrac{2}{a} \cdot \dfrac{4a^2}{3} = \dfrac{2}{\overset{}{\underset{1}{a}}} \cdot \dfrac{4\overset{a}{a^2}}{3} = \dfrac{8a}{3}$$

Answer $\dfrac{8a}{3}$ $(a \neq 0)$

EXAMPLE 2

Simplify: $\dfrac{\frac{b}{10} + \frac{1}{2}}{\frac{b^2}{25} - 1}$

Solution

METHOD I	METHOD 2
The least common multiple of 10, 2, and 25 is 50.	$\dfrac{\frac{b}{10} + \frac{1}{2}}{\frac{b^2}{25} - 1} = \dfrac{\frac{b}{10} + \frac{5}{10}}{\frac{b^2}{25} - \frac{25}{25}}$

METHOD I:

$$\dfrac{\frac{b}{10} + \frac{1}{2}}{\frac{b^2}{25} - 1} = \dfrac{\frac{b}{10} + \frac{1}{2}}{\frac{b^2}{25} - 1} \cdot \dfrac{50}{50}$$

$$= \dfrac{5b + 25}{2b^2 - 50}$$

$$= \dfrac{5(b + 5)}{2(b + 5)(b - 5)}$$

$$= \dfrac{5}{2(b - 5)}$$

METHOD 2:

$$= \dfrac{\frac{b + 5}{10}}{\frac{b^2 - 25}{25}}$$

$$= \dfrac{b + 5}{10} \div \dfrac{b^2 - 25}{25}$$

$$= \dfrac{\overset{1}{b + 5}}{\underset{2}{10}} \cdot \dfrac{\overset{5}{25}}{\underset{1}{(b + 5)(b - 5)}}$$

$$= \dfrac{5}{2(b - 5)}$$

Answer $\dfrac{5}{2(b - 5)}$ $(b \neq -5, 5)$

Exercises

Writing About Mathematics

1. For what values of a is $\left(1 - \frac{1}{a}\right) \div \left(1 - \frac{1}{a^2}\right) = \dfrac{1 - \frac{1}{a}}{1 - \frac{1}{a^2}}$ undefined? Explain your answer.

2. Bebe said that since each of the denominators in the complex fraction $\dfrac{\frac{d}{4} + \frac{3}{5}}{2 - \frac{d^2}{2}}$ is a non-zero constant, the fraction is defined for all values of d. Do you agree with Bebe? Explain why or why not.

Developing Skills

In 3–20, simplify each complex rational expression. In each case, list any values of the variables for which the fractions are not defined.

3. $\dfrac{\frac{3}{3}}{\frac{3}{4}}$

4. $\dfrac{\frac{2}{5}}{4}$

5. $\dfrac{\frac{7}{8}}{1\frac{3}{4}}$

6. $\dfrac{\frac{2}{x}}{\frac{1}{2x}}$

7. $\dfrac{\frac{1}{6}+\frac{1}{4}}{\frac{1}{3}+\frac{1}{2}}$

8. $\dfrac{1+\frac{1}{a}}{a+1}$

9. $\dfrac{2-\frac{2}{d}}{\frac{1}{d}-1}$

10. $\dfrac{b-\frac{1}{b}}{\frac{1}{b}-1}$

11. $\dfrac{3-\frac{3}{b}}{b-1}$

12. $\dfrac{y+\frac{1}{2}}{2y+1}$

13. $\dfrac{4y-\frac{1}{y}}{y-\frac{1}{2}}$

14. $\dfrac{\frac{1}{2x}+\frac{1}{3x}}{\frac{1}{x^2}}$

15. $\dfrac{a-\frac{49}{a}}{a-9+\frac{14}{a}}$

16. $\dfrac{3-\frac{9}{x}}{x-8+\frac{15}{x}}$

17. $\dfrac{1+\frac{3}{b}+\frac{2}{b^2}}{1-\frac{1}{b^2}}$

18. $\dfrac{1+\frac{1}{y}-\frac{6}{y^2}}{1+\frac{11}{y}+\frac{24}{y^2}}$

19. $\dfrac{\frac{1}{a}-1}{1-\frac{1}{a}}$

20. $\dfrac{5-\frac{45}{a^2}}{\frac{3}{a}-1}$

In 21–24, simplify each expression. In each case, list any values of the variables for which the fractions are not defined.

21. $\dfrac{3}{2x}-\dfrac{1+\frac{1}{x}}{x+1}$

22. $\dfrac{\frac{a}{4}+\frac{7a}{8}}{\frac{6a^2}{5}-\frac{3a^2}{10}}+\dfrac{3}{a}$

23. $\left(\dfrac{3}{a}+\dfrac{5}{a^2}\right)\div\left(\dfrac{10}{a}+6\right)+\dfrac{3}{4}$

24. $\left(6+\dfrac{12}{b}\right)\div\left(3b-\dfrac{12}{b}\right)+\dfrac{b}{2-b}$

2-7 SOLVING RATIONAL EQUATIONS

An equation in which there is a fraction in one or more terms can be solved in different ways. However, in each case, the end result must be an equivalent equation in which the variable is equal to a constant.

For example, solve: $\frac{x}{4} - 2 = \frac{x}{2} + \frac{1}{2}$. This is an equation with rational coefficients and could be written as $\frac{1}{4}x - 2 = \frac{1}{2}x + \frac{1}{2}$. There are three possible methods of solving this equation.

METHOD 1	
Work with the fractions as given. Combine terms containing the variable on one side of the equation and constants on the other. Find common denominators to add or subtract fractions.	$\frac{x}{4} - 2 = \frac{x}{2} + \frac{1}{2}$ $\frac{x}{4} - \frac{x}{2} = \frac{1}{2} + 2$ $\frac{x}{4} - \frac{2x}{4} = \frac{1}{2} + \frac{4}{2}$ $-\frac{x}{4} = \frac{5}{2}$ $x = \frac{5}{2}(-4) = -\frac{20}{2} = -10$

METHOD 2	
Rewrite the equation without fractions. Multiply both sides of the equation by the least common denominator. In this case, the LCD is 4.	$\frac{x}{4} - 2 = \frac{x}{2} + \frac{1}{2}$ $4\left(\frac{x}{4} - 2\right) = 4\left(\frac{x}{2} + \frac{1}{2}\right)$ $x - 8 = 2x + 2$ $-x = 10$ $x = -10$

METHOD 3	
Rewrite each side of the equation as a ratio. Use the fact that the product of the means is equal to the product of the extremes.	$\frac{x}{4} - 2 = \frac{x}{2} + \frac{1}{2}$ $\frac{x}{4} - \frac{8}{4} = \frac{x}{2} + \frac{1}{2}$ $\frac{x - 8}{4} = \frac{x + 1}{2}$ $4(x + 1) = 2(x - 8)$ $4x + 4 = 2x - 16$ $2x = -20$ $x = -10$

These same procedures can also be used for a **rational equation**, an equation in which the variable appears in one or more denominators.

EXAMPLE I

Solve the equation $\frac{1}{x} + \frac{1}{3} = \frac{3}{2x} - 1$.

Solution   Use Method 2.

How to Proceed

(1) Write the equation:
$$\frac{1}{x} + \frac{1}{3} = \frac{3}{2x} - 1$$

(2) Multiply both sides of the equation by the least common denominator, $6x$:
$$6x\left(\frac{1}{x} + \frac{1}{3}\right) = 6x\left(\frac{3}{2x} - 1\right)$$

(3) Simplify:
$$\frac{6x}{x} + \frac{6x}{3} = \frac{18x}{2x} - 6x$$
$$6 + 2x = 9 - 6x$$

(4) Solve for x:
$$8x = 3$$
$$x = \frac{3}{8}$$

Check

$$\frac{1}{x} + \frac{1}{3} = \frac{3}{2x} - 1$$
$$\frac{1}{\frac{3}{8}} + \frac{1}{3} \overset{?}{=} \frac{3}{2\left(\frac{3}{8}\right)} - 1$$
$$\left(1 \times \frac{8}{3}\right) + \frac{1}{3} \overset{?}{=} \left(3 \times \frac{4}{3}\right) - 1$$
$$\frac{8}{3} + \frac{1}{3} \overset{?}{=} 4 - 1$$
$$\frac{9}{3} \overset{?}{=} 3$$
$$3 = 3 ✔$$

Answer $x = \frac{3}{8}$

When solving a rational equation, it is important to check that the fractional terms are defined for each root, or solution, of the equation. Any solution for which the equation is undefined is called an **extraneous root**.

EXAMPLE 2

Solve for a: $\frac{a - 2}{a} = \frac{a + 2}{2a}$

Solution   Since this equation is a proportion, we can use that the product of the means is equal to the product of the extremes. Check the roots.

$$\frac{a - 2}{a} = \frac{a + 2}{2a}$$
$$2a(a - 2) = a(a + 2)$$
$$2a^2 - 4a = a^2 + 2a$$
$$a^2 - 6a = 0$$
$$a(a - 6) = 0$$
$$a = 0 \quad | \quad a - 6 = 0$$
$$\qquad\qquad a = 6$$

Check: $a = 0$

$$\frac{a - 2}{a} = \frac{a + 2}{2a}$$
$$\frac{0 - 2}{0} \overset{?}{=} \frac{0 + 2}{2(0)}$$

Each side of the equation is undefined for $a = 0$. Therefore, $a = 0$ is not a root.

Check: $a = 6$

$$\frac{a - 2}{a} = \frac{a + 2}{2a}$$
$$\frac{6 - 2}{6} \overset{?}{=} \frac{6 + 2}{2(6)}$$
$$\frac{4}{6} \overset{?}{=} \frac{8}{12}$$
$$\frac{2}{3} = \frac{2}{3} ✔$$

Answer $a = 6$

EXAMPLE 3

Solve: $\dfrac{x}{x + 2} = \dfrac{3}{x} + \dfrac{4}{x(x + 2)}$

Solution Use Method 2 to rewrite the equation without fractions. The least common denominator is $x(x + 2)$.

$$\frac{x}{x + 2} = \frac{3}{x} + \frac{4}{x(x + 2)}$$

$$x(x + 2)\left(\frac{x}{x + 2}\right) = x(x + 2)\left(\frac{3}{x}\right) + x(x + 2)\left(\frac{4}{x(x + 2)}\right)$$

$$x(x + 2)\left(\frac{x}{x + 2}\right) = x(x + 2)\left(\frac{3}{x}\right) + x(x + 2)\left(\frac{4}{x(x + 2)}\right)$$

$$x^2 = 3x + 6 + 4$$

$$x^2 - 3x - 10 = 0$$

$$(x - 5)(x + 2) = 0$$

$$x - 5 = 0 \quad | \quad x + 2 = 0$$

$$x = 5 \quad | \quad x = -2$$

Check the roots:

Check: x = 5	*Check: x = − 2*
$\dfrac{x}{x + 2} = \dfrac{3}{x} + \dfrac{4}{x(x + 2)}$	$\dfrac{x}{x + 2} = \dfrac{3}{x} + \dfrac{4}{x(x + 2)}$
$\dfrac{5}{5 + 2} \overset{?}{=} \dfrac{3}{5} + \dfrac{4}{5(5 + 2)}$	$\dfrac{-2}{-2 + 2} \overset{?}{=} \dfrac{3}{-2} + \dfrac{4}{-2(-2 + 2)}$
$\dfrac{5}{7} \times \dfrac{5}{5} \overset{?}{=} \dfrac{3}{5} \times \dfrac{7}{7} + \dfrac{4}{35}$	$\dfrac{-2}{0} \overset{?}{=} -\dfrac{3}{2} + \dfrac{4}{-2(0)}$
$\dfrac{25}{35} \overset{?}{=} \dfrac{21}{35} + \dfrac{4}{35}$	Since $x = -2$ leads to a statement that is undefined, -2 is not a root of the equation.
$\dfrac{25}{35} = \dfrac{25}{35} ✔$	

Since $x = 5$ leads to a true statement, 5 is a root of the equation.

Answer 5 is the only root of $\dfrac{x}{x + 2} = \dfrac{3}{x} + \dfrac{4}{x(x + 2)}$.

It is important to check each root obtained by multiplying both members of the given equation by an expression that contains the variable. The derived equation may not be equivalent to the given equation.

EXAMPLE 4

Solve and check: $0.2y + 4 = 5 - 0.05y$

Solution   The coefficients 0.2 and 0.05 are decimal fractions ("2 *tenths*" and "5 *hundredths*"). The denominator of 0.2 is 10 and the denominator of 0.05 is 100. Therefore, the least common denominator is 100.

	Check
$0.2y + 4 = 5 - 0.05y$	$0.2y + 4 = 5 - 0.05y$
$100(0.2y + 4) = 100(5 - 0.05y)$	$0.2(4) + 4 \overset{?}{=} 5 - 0.05(4)$
$20y + 400 = 500 - 5y$	$0.8 + 4 \overset{?}{=} 5 - 0.2$
$25y = 100$	$4.8 = 4.8$ ✔
$y = 4$	

Answer $y = 4$

EXAMPLE 5

On his way home from college, Daniel traveled 15 miles on local roads and 90 miles on the highway. On the highway, he traveled 30 miles per hour faster than on local roads. If the trip took 2 hours, what was Daniel's rate of speed on each part of the trip?
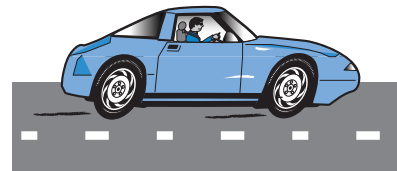
Solution   Use $\frac{\text{distance}}{\text{rate}} = $ time to represent the time for each part of the trip.

Let $x = $ rate on local roads. Then $\frac{15}{x} = $ time on local roads.

Let $x + 30 = $ rate on the highway. Then $\frac{90}{x + 30} = $ time on the highway.

The total time is 2 hours.

$$\frac{15}{x} + \frac{90}{x + 30} = 2$$
$$x(x + 30)\left(\frac{15}{x}\right) + x(x + 30)\left(\frac{90}{x + 30}\right) = 2x(x + 30)$$
$$15(x + 30) + 90x = 2x^2 + 60x$$
$$15x + 450 + 90x = 2x^2 + 60x$$
$$0 = 2x^2 - 45x - 450$$

Recall that a trinomial can be factored by writing the middle term as the sum of two terms whose product is the product of the first and last term.

$$2x^2(-450) = -900x^2$$

Find the factors of this product whose sum is the middle term $-45x$.

$$15x + (-60x) = -45x$$

Write the trinomial with four terms, using this pair of terms in place of $-45x$, and factor the trinomial.

$$0 = 2x^2 - 45x - 450$$
$$0 = 2x^2 + 15x - 60x - 450$$
$$0 = x(2x + 15) - 30(2x + 15)$$
$$0 = (2x + 15)(x - 30)$$

$2x + 15 = 0$	$x - 30 = 0$
$2x = -15$	$x = 30$
$x = -\frac{15}{2}$	

Reject the negative root since a rate cannot be a negative number. Use $x = 30$.

On local roads, Daniel's rate is 30 miles per hour and his time is $\frac{15}{30} = \frac{1}{2}$ hour.

On the highway, Daniel's rate is $30 + 30$ or 60 miles per hour. His time is $\frac{90}{60} = \frac{3}{2}$ hours. His total time is $\frac{1}{2} + \frac{3}{2} = \frac{4}{2} = 2$ hours.

Answer Daniel drove 30 mph on local roads and 60 mph on the highway.

Exercises

Writing About Mathematics

1. Samantha said that the equation $\frac{a - 2}{a} = \frac{a + 2}{2a}$ in Example 2 could be solved by multiplying both sides of the equation by $2a$. Would Samantha's solution be the same as the solution obtained in Example 2? Explain why or why not.

2. Brianna said that $\frac{3}{x - 2} = \frac{5}{x + 2}$ is a rational equation but $\frac{x - 2}{3} = \frac{x + 2}{5}$ is not. Do you agree with Brianna? Explain why or why not.

Developing Skills

In 3–20, solve each equation and check.

3. $\frac{1}{4}a + 8 = \frac{1}{2}a$

4. $\frac{3}{4}x = 14 - x$

5. $\frac{x + 2}{5} = \frac{x - 2}{3}$

6. $\frac{x}{5} - \frac{x}{10} = 7$

7. $\frac{2x}{3} + 1 = \frac{3x}{4}$

8. $x - 0.05x = 19$

9. $0.4x + 8 = 0.5x$

10. $0.2a = 0.05a + 3$

11. $1.2b - 3 = 7 - 0.05b$

12. $\frac{x}{x + 5} = \frac{2}{3}$

13. $\frac{7}{2x - 3} = \frac{4}{x}$

14. $\frac{3}{a} + \frac{1}{2} = \frac{11}{5a}$

15. $18 - \frac{4}{b} = 10$

16. $\frac{x}{2} = \frac{3}{2x + 1}$

17. $1 = \frac{5}{x + 3} + \frac{5}{(x + 2)(x + 3)}$

18. $\frac{a - 1}{4} = \frac{8}{a + 3}$

19. $\frac{4}{y + 2} = 1 - \frac{8}{y(y + 2)}$

20. $\frac{4}{3b - 2} - \frac{7}{3b + 2} = \frac{1}{9b^2 - 4}$

Applying Skills

21. Last week, the ratio of the number of hours that Joseph worked to the number of hours that Nicole worked was 2 : 3. This week Joseph worked 4 hours more than last week and Nicole worked twice as many hours as last week. This week the ratio of the hours Joseph worked to the number of hours Nicole worked is 1 : 2. How many hours did each person work each week?

22. Anthony rode his bicycle to his friend's house, a distance of 1 mile. Then his friend's mother drove them to school, a distance of 12 miles. His friend's mother drove at a rate that is 25 miles per hour faster than Anthony rides his bike. If it took Anthony $\frac{3}{5}$ of an hour to get to school, at what average rate does he ride his bicycle? (Use $\frac{\text{distance}}{\text{rate}} = $ time for each part of the trip to school.)

23. Amanda drove 40 miles. Then she increased her rate of speed by 10 miles per hour and drove another 40 miles to reach her destination. If the trip took $1\frac{4}{5}$ hours, at what rate did Amanda drive?

24. Last week, Emily paid $8.25 for x pounds of apples. This week she paid $9.50 for $(x + 1)$ pounds of apples. The price per pound was the same each week. How many pounds of apples did Emily buy each week and what was the price per pound? (Use $\frac{\text{total cost}}{\text{number of pounds}} = $ cost per pound for each week.)

2-8 SOLVING RATIONAL INEQUALITIES

Inequalities are usually solved with the same procedures that are used to solve equations. For example, we can solve this equation and this inequality by using the same steps.

$$\frac{1}{4}x + \frac{1}{8} = \frac{1}{3}x + \frac{3}{8}$$
$$24\left(\frac{1}{4}x\right) + 24\left(\frac{1}{8}\right) = 24\left(\frac{1}{3}x\right) + 24\left(\frac{3}{8}\right)$$
$$6x + 3 = 8x + 9$$
$$-2x = 6$$
$$x = -3$$

$$\frac{1}{4}x + \frac{1}{8} < \frac{1}{3}x + \frac{3}{8}$$
$$24\left(\frac{1}{4}x\right) + 24\left(\frac{1}{8}\right) < 24\left(\frac{1}{3}x\right) + 24\left(\frac{3}{8}\right)$$
$$6x + 3 < 8x + 9$$
$$-2x < 6$$
$$x > -3$$

All steps leading to the solution of this equation and this inequality are the same, but special care must be used when multiplying or dividing an inequality. Note that when we multiplied the inequality by 24, a positive number, the order of the inequality remained unchanged. In the last step, when we divided the inequality by -2, a negative number, the order of the inequality was reversed.

When we solve an inequality that has a variable expression in the denominator by multiplying both sides of the inequality by the variable expression, we must consider two possibilities: the variable represents a positive number or the

variable represents a negative number. (The expression cannot equal zero as that would make the fraction undefined.)

For example, to solve $2 - \frac{x}{x + 1} > 3 - \frac{9}{x + 1}$, we multiply both sides of the inequality by $x + 1$. When $x + 1$ is positive, the order of the inequality remains unchanged. When $x + 1$ is negative, the order of the inequality is reversed.

If $x + 1 > 0$, then $x > -1$.	If $x + 1 < 0$, then $x < -1$.
$2 - \frac{x}{x + 1} > 3 - \frac{9}{x + 1}$	$2 - \frac{x}{x + 1} > 3 - \frac{9}{x + 1}$
$\frac{9}{x + 1} - \frac{x}{x + 1} > 3 - 2$	$\frac{9}{x + 1} - \frac{x}{x + 1} > 3 - 2$
$(x + 1)\frac{9 - x}{x + 1} > (x + 1)(1)$	$(x + 1)\frac{9 - x}{x + 1} < (x + 1)(1)$
$9 - x > x + 1$	$9 - x < x + 1$
$\frac{-2x}{-2} < \frac{-8}{-2}$	$\frac{-2x}{-2} > \frac{-8}{-2}$
$x < 4$	$x > 4$
Therefore, $x > -1$ and $x < 4$ or $-1 < x < 4$.	There are no values of x such that $x < -1$ and $x > 4$.

The solution set of the equation $2 - \frac{x}{x + 1} > 3 - \frac{9}{x + 1}$ is $\{x : -1 < x < 4\}$.

An alternative method of solving this inequality is to use the corresponding equation.

STEP 1. Solve the corresponding equation.

$$2 - \frac{x}{x + 1} = 3 - \frac{9}{x + 1}$$
$$(x + 1)(2) - (x + 1)\frac{x}{x + 1} = (x + 1)(3) - (x + 1)\frac{9}{x + 1}$$
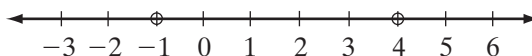$$2x + 2 - x = 3x + 3 - 9$$
$$-2x = -8$$
$$x = 4$$

STEP 2. Find any values of x for which the equation is undefined.

Terms $\frac{x}{x + 1}$ and $\frac{9}{x + 1}$ are undefined when $x = -1$.

STEP 3. To find the solutions of the corresponding inequality, divide the number line into three intervals using the solution to the equality, 4, and the value of x for which the equation is undefined, -1.
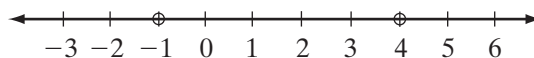
Choose a value from each section and substitute that value into the inequality.

Let $x = -2$:	**Let $x = 0$:**	**Let $x = 5$:**

Let $x = -2$:

$$2 - \frac{x}{x+1} > 3 - \frac{9}{x+1}$$

$$2 - \frac{-2}{-2+1} \overset{?}{>} 3 - \frac{9}{-2+1}$$

$$2 - (2) \overset{?}{>} 3 - (-9)$$

$$0 \not> 12 \; ✗$$

Let $x = 0$:

$$2 - \frac{x}{x+1} > 3 - \frac{9}{x+1}$$

$$2 - \frac{0}{0+1} \overset{?}{>} 3 - \frac{9}{0+1}$$

$$2 - (0) \overset{?}{>} 3 - (9)$$

$$2 > -6 \; ✔$$

Let $x = 5$:

$$2 - \frac{x}{x+1} > 3 - \frac{9}{x+1}$$

$$2 - \frac{5}{5+1} \overset{?}{>} 3 - \frac{9}{5+1}$$

$$2 - \frac{5}{6} \overset{?}{>} 3 - \frac{9}{6}$$

$$\frac{7}{6} \not> \frac{9}{6} \; ✗$$

The inequality is true for values in the interval $-1 < x < 4$ and false for all other values.

EXAMPLE I

Solve for a: $2 - \frac{3}{a} > \frac{5}{a}$

Solution   **METHOD I**

Multiply both sides of the equation by a. The sense of the inequality will remain unchanged when $a > 0$ and will be reversed when $a < 0$.

Let $a > 0$:

$$a\left(2 - \frac{3}{a}\right) > a\left(\frac{5}{a}\right)$$
$$2a - 3 > 5$$
$$2a > 8$$
$$a > 4$$
$$a > 0 \text{ and } a > 4 \to a > 4$$

Let $a < 0$:

$$a\left(2 - \frac{3}{a}\right) < a\left(\frac{5}{a}\right)$$
$$2a - 3 < 5$$
$$2a < 8$$
$$a < 4$$
$$a < 0 \text{ and } a < 4 \to a < 0$$
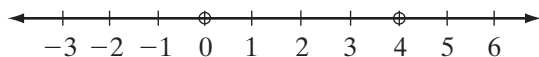
The solution set is $\{a : a < 0 \text{ or } a > 4\}$.

METHOD 2

Solve the corresponding equation for a.

$$2 - \frac{3}{a} = \frac{5}{a}$$
$$a\left(2 - \frac{3}{a}\right) = a\left(\frac{5}{a}\right)$$
$$2a - 3 = 5$$
$$2a = 8$$
$$a = 4$$

Partition the number line using the solution to the equation, $a = 4$, and the value of a for which the equation is undefined, $a = 0$. Check in the inequality a representative value of a from each interval of the graph.

$$\longleftarrow \underset{-3\ -2\ -1\ \ 0\ \ \ 1\ \ \ 2\ \ \ 3\ \ \ 4\ \ \ 5\ \ \ 6}{\rule{0pt}{0pt}} \longrightarrow$$

Let $a = -1$:

$2 - \frac{3}{a} > \frac{5}{a}$

$2 - \frac{3}{-1} \overset{?}{>} \frac{5}{-1}$

$2 + 3 \overset{?}{>} -5$

$5 > -5$ ✔

Let $a = 1$:

$2 - \frac{3}{a} > \frac{5}{a}$

$2 - \frac{3}{1} \overset{?}{>} \frac{5}{1}$

$2 - 3 \overset{?}{>} 5$

$-1 \not> 5$ ✗

Let $a = 5$:

$2 - \frac{3}{a} > \frac{5}{a}$

$2 - \frac{3}{5} \overset{?}{>} \frac{5}{5}$

$\frac{10}{5} - \frac{3}{5} \overset{?}{>} 1$

$\frac{7}{5} > 1$ ✔

Answer $\{a : a < 0 \text{ or } a > 4\}$

Exercises

Writing About Mathematics

1. When the equation $2 - \frac{3}{b} = \frac{5}{b + 2}$ is solved for b, the solutions are -1 and 3. Explain why the number line must be separated into five segments by the numbers $-2, -1, 0,$ and 3 in order to check the solution set of the inequality $2 - \frac{3}{b} > \frac{5}{b + 2}$.

2. What is the solution set of $\frac{|x|}{x} < 0$? Justify your answer.

Developing Skills

In 3–14, solve and check each inequality.

3. $\frac{a}{4} > \frac{a}{2} + 6$

4. $\frac{y - 3}{5} < \frac{y + 2}{10}$

5. $\frac{3b - 4}{8} < \frac{4b - 3}{4}$

6. $\frac{2 - d}{7} > \frac{d - 2}{5}$

7. $\frac{a + 1}{4} - 2 > 11 - \frac{a}{6}$

8. $\frac{7}{2x} - \frac{2}{x} > \frac{3}{2}$

9. $5 - \frac{7}{y} < 2 + \frac{5}{y}$

10. $3 - \frac{2}{a + 1} < 5$

11. $\frac{4x}{x - 4} + 2 < \frac{2}{x - 4}$

12. $\frac{2}{x} + \frac{3}{x} < 10$

13. $\frac{x}{x + 5} - \frac{1}{x + 5} > 4$

14. $3 - \frac{9}{a + 1} > 5 - \frac{1}{a + 1}$

CHAPTER SUMMARY

A **rational number** is an element of the set of numbers of the form $\frac{a}{b}$ when a and b are integers and $b \neq 0$. For every rational number $\frac{a}{b}$ that is not equal to zero there is a **multiplicative inverse** or **reciprocal** $\frac{b}{a}$ such that $\frac{a}{b} \cdot \frac{b}{a} = 1$.

A number is a rational number if and only if it can be written as an infinitely repeating decimal.

A **rational expression** is the quotient of two polynomials. A rational expression has no meaning when the denominator is zero.

A rational expression is in **simplest form** or **reduced to lowest terms** when there is no factor of the numerator that is also a factor of the denominator except 1 and -1.

If $\frac{a}{b}$ and $\frac{c}{d}$ are two rational numbers with $b \neq 0$ and $d \neq 0$:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc} \ (c \neq 0)$$

$$\frac{a}{b} + \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{d} + \frac{c}{d} \cdot \frac{b}{b} = \frac{ad}{bd} + \frac{cb}{bd} = \frac{ad + cb}{bd}$$

A **ratio** is the comparison of two numbers by division. The ratio of a to b can be written as $\frac{a}{b}$ or as $a : b$ when $b \neq 0$.

An equation that states that two ratios are equal is called a **proportion**. In the proportion $a : b = c : d$ or $\frac{a}{b} = \frac{c}{d}$, the first and last terms, a and d, are called the **extremes**, and the second and third terms, b and c, are the **means**.

In any proportion, the product of the means is equal to the product of the extremes.

A **complex rational expression** has a rational expression in the numerator, the denominator or both. Complex rational expressions can be simplified by multiplying numerator and denominator by a common multiple of the denominators in the numerator and denominator.

A **rational equation**, an equation in which the variable appears in one or more denominators, can be simplified by multiplying both members by a common multiple of the denominators.

When a rational inequality is multiplied by the least common multiple of the denominators, two cases must be considered: when the least common multiple is positive and when the least common multiple is negative.

VOCABULARY

2-1 Rational number • Multiplicative inverse • Reciprocal

2-2 Rational expression • Simplest form • Lowest terms

2-4 Least common denominator (LCD) • Least common multiple (LCM)

2-5 Ratio • Ratio in simplest form • Proportion • Extremes • Means

2-6 Complex fraction • Complex rational expression

2-7 Rational equation • Extraneous root

REVIEW EXERCISES

1. What is the multiplicative inverse of $\frac{7}{2}$?

2. For what values of b is the rational expression $\frac{2(b + 1)}{b(b^2 - 1)}$ undefined?

3. Write $\frac{5}{12}$ as an infinitely repeating decimal.

In 4–15, perform the indicated operations, express the answer in simplest form, and list the values of the variables for which the fractions are undefined.

4. $\frac{3}{5ab^2} \cdot \frac{10a^2b}{9}$

5. $\frac{2}{5a} + \frac{1}{4a}$

6. $\frac{3x + 12}{4x} \cdot \frac{2}{x^2 - 16}$

7. $\frac{a^2 + 2a}{5a} \div \frac{a^2 + 7a + 10}{a^2}$

8. $\frac{2}{a + 1} + \frac{3}{a^2 - 1}$

9. $\frac{y}{y - 3} - \frac{18}{y^2 - 9}$

10. $\frac{d^2 + 3d - 18}{4d} \div \frac{2d + 12}{8}$

11. $\frac{3}{5b} \cdot \frac{5b + 10}{6} - \frac{1}{b}$

12. $\frac{1}{a + 2} + \frac{a}{a^2 + a} \div \frac{a}{a + 1}$

13. $\left(\frac{1}{a - 4} + \frac{3}{4 - a}\right) \cdot \frac{16 - a^2}{2}$

14. $\left(1 + \frac{1}{x + 1}\right) \cdot \left(1 + \frac{3}{x^2 - 4}\right)$

15. $\left(x - \frac{1}{x}\right) \div (1 - x)$

In 16–19, simplify each complex rational expression and list the values of the variables (if any) for which the fractions are undefined.

16. $\dfrac{1 + \frac{1}{2}}{2 + \frac{1}{4}}$

17. $\dfrac{\frac{a + b}{b}}{1 + \frac{b}{a}}$

18. $\dfrac{\frac{x}{12} - \frac{1}{2}}{\frac{x^2}{12} - 3}$

19. $\dfrac{a - 1 - \frac{20}{a}}{1 - \frac{16}{a^2}}$

In 20–27, solve and check each equation or inequality.

20. $\frac{a}{7} + \frac{3}{14} = \frac{5a}{2}$

21. $\frac{x}{5} + 9 = 12 - \frac{x}{4}$

22. $\frac{3}{y} + \frac{1}{2} = \frac{6}{y}$

23. $\frac{2x}{x - 2} = \frac{3x}{x + 1}$

24. $\frac{x}{x - 3} + \frac{6}{x(x - 3)} = \frac{5}{x - 3}$

25. $\frac{2d + 1}{3} = \frac{2d + 2}{d} - 2$

26. $\frac{1}{x} + 3 > \frac{7}{x}$

27. $\frac{1}{2x + 1} - 2 \leq 8$

28. The ratio of boys to girls in the school chorus was 4 : 5. After three more boys joined the chorus, the ratio of boys to girls is 9 : 10. How many boys and how many girls are there now in the chorus?

29. Last week Stephanie spent $10.50 for cans of soda. This week, at the same cost per can, she bought three fewer cans and spent $8.40. How many cans of soda did she buy each week and what was the cost per can?

30. On a recent trip, Sarah traveled 80 miles at a constant rate of speed. Then she encountered road work and had to reduce her speed by 15 miles per hour for the last 30 miles of the trip. The trip took 2 hours. What was Sarah's rate of speed for each part of the trip?

31. The areas of two rectangles are 208 square feet and 182 square feet. The length of the larger is 2 feet more than the length of the smaller. If the rectangles have equal widths, find the dimensions of each rectangle. $\left(\text{Use } \frac{\text{area}}{\text{length}} = \text{width.}\right)$

Exploration

The early Egyptians wrote fractions as the sum of **unit fractions** (the reciprocals of the counting numbers) with no reciprocal repeated. For example:

$$\tfrac{3}{4} = \tfrac{2}{4} + \tfrac{1}{4} = \tfrac{1}{2} + \tfrac{1}{4} \quad \text{and} \quad \tfrac{2}{3} = \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{12}$$

Of course the Egyptians used hieroglyphs instead of the numerals familiar to us today. Whole numbers were represented as follows.

To write unit fractions, the symbol ⬭ was drawn over the whole number hieroglyph. For example:

$$\overset{\frown}{|||} = \tfrac{1}{3} \quad \text{and} \quad \overset{\frown}{\cap\cap} = \tfrac{1}{20}$$

1. Show that for $n \neq 0$, $\dfrac{1}{n} = \dfrac{1}{n+1} + \dfrac{1}{n(n+1)}$.

2. Write $\tfrac{3}{4}$ and $\tfrac{2}{3}$ using Egyptian fractions.

3. Write each of the fractions as the sum of unit fractions. (*Hint:* Use the results of Exercise 1.)

   a. $\tfrac{2}{5}$ **b.** $\tfrac{7}{10}$ **c.** $\tfrac{7}{12}$ **d.** $\tfrac{23}{24}$ **e.** $\tfrac{11}{18}$

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. Which of the following is *not* true in the set of integers?
  (1) Addition is commutative.
  (2) Every integer has an additive inverse.
  (3) Multiplication is distributive over addition.
  (4) Every non-zero integer has a multiplicative inverse.

2. $2 - |3 - 5|$ is equal to
  (1) 0 (2) 2 (3) 6 (4) 4

3. The sum of $3a^2 - 5a$ and $a^2 + 7a$ is
  (1) $3 + 2a$ (2) $3a^2 + 2a$ (3) $4a^2 + 2a$ (4) $6a^2$

4. In simplest form, $2x(x + 5) - (7x + 1)$ is equal to
  (1) $2x^2 + 3x + 1$ (3) $2x^2 + 12x + 1$
  (2) $2x^2 + 3x - 1$ (4) $2x^2 - 7x + 4$

5. The factors of $2x^2 - x - 6$ are
  (1) $(2x - 3)(x + 2)$ (3) $(2x + 3)(x - 2)$
  (2) $(2x + 2)(x - 3)$ (4) $(2x - 3)(x - 2)$

6. The roots of the equation $x^2 - 7x + 10 = 0$ are
  (1) 2 and 5 (3) -7 and 10
  (2) -2 and -5 (4) -5 and 2

7. For $a \neq 0, 2, -2$, the fraction $\dfrac{\frac{2}{a} - 1}{a - \frac{4}{a}}$ is equal to

  (1) $\dfrac{1}{a + 2}$ (3) $\dfrac{1}{a - 2}$
  (2) $-\dfrac{1}{a + 2}$ (4) $-\dfrac{1}{a - 2}$

8. In simplest form, the quotient $\dfrac{b}{2b - 1} \div \dfrac{3b}{8b - 4}$ equals
  (1) $\frac{3}{4}$ (3) $\dfrac{3b^2}{4(2b - 1)^2}$
  (2) $\frac{4}{3}$ (4) $\dfrac{7a}{4(2b - 1)}$

9. For what values of a is the fraction $\dfrac{2a + 4}{a^2 - 2a - 35}$ undefined?
  (1) $-2, -5, 7$ (2) $2, 5, -7$ (3) $-5, 7$ (4) $5, -7$

10. The solution set of the equation $|2x - 1| = 7$ is
  (1) $\{-3, 4\}$ (2) $\varnothing$ (3) $\{-3\}$ (4) $\{4\}$

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. Solve and check: $|7 - 2x| \leq 3$

12. Perform the indicated operations and write the answer in simplest form:

$$\frac{3}{a+5} - \frac{a-3}{5} \div \frac{a^2-9}{15}$$

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. Write the following product in simplest form and list the values of x for which it is undefined: $\frac{5x+5}{x^2-1} \cdot \frac{x^2-x}{15}$

14. The length of a rectangle is 12 meters longer than half the width. The area of the rectangle is 90 square meters. Find the dimensions of the rectangle.

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. Find the solution set and check: $2x^2 - 5x > 7$

16. Diego had traveled 30 miles at a uniform rate of speed when he encountered construction and had to reduce his speed to one-third of his original rate. He continued at this slower rate for 10 miles. If the total time for these two parts of the trip was one hour, how fast did he travel at each rate?