CHAPTER

11

THE GEOMETRY OF THREE DIMENSIONS

Bonaventura Cavalieri (1598–1647) was a follower of Galileo and a mathematician best known for his work on areas and volumes. In this aspect, he proved to be a forerunner of the development of integral calculus. His name is associated with *Cavalieri's Principle* which is a fundamental principle for the determination of the volume of a solid.

Cavalieri's Principle can be stated as follows:

- Given two geometric solids and a plane, if every plane parallel to the given plane that intersects both solids intersects them in surfaces of equal areas, then the volumes of the two solids are equal.

This means that two solids have equal volume when their corresponding cross-sections are in all cases equal.

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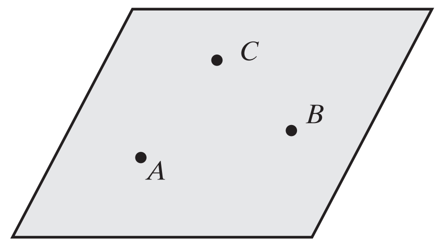
11-1 POINTS, LINES, AND PLANES

In this text, we have been studying points and lines in a plane, that is, the geometry of two dimensions. But the world around us is three-dimensional. The geometry of three dimensions is called **solid geometry**. We begin this study with some postulates that we can accept as true based on our observations.

We know that two points determine a line. How many points are needed to determine a plane? A table or chair that has four legs will sometimes be unsteady on a flat surface. But a tripod or a stool with three legs always sits firmly. This observation suggests the following postulate.

Postulate 11.1 There is one and only one plane containing three non-collinear points.

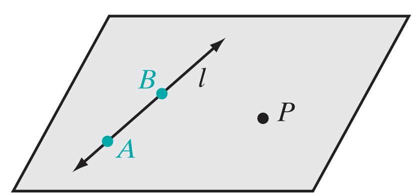
For a set of the three non-collinear points that determine a plane, each pair of points determines a line and all of the points on that line are points of the plane.



Postulate 11.2 A plane containing any two points contains all of the points on the line determined by those two points.

These two postulates make it possible for us to prove the following theorems.

Theorem 11.1 There is exactly one plane containing a line and a point not on the line.



Given Line l and point P not on l.

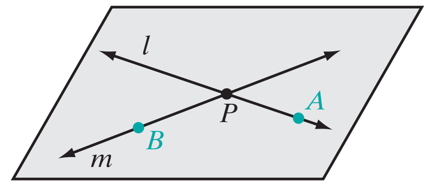
Prove There is exactly one plane containing l and P.

Proof Choose two points A and B on line l. The three points, A, B, and P, determine one and only one plane. If the two points A and B on line l are on the plane, then all of the points of l are on the plane, that is, the plane contains line l. Therefore, there is exactly one plane that contains the given line and point.

Theorem 11.2 If two lines intersect, then there is exactly one plane containing them.

This theorem can be stated in another way:

Theorem 11.2 Two intersecting lines determine a plane.



Given Lines l and m intersecting at point P.

Prove There is exactly one plane containing l and m.

Proof Choose two points, A on line l and B on line m. The three points, A, B, and P, determine one and only one plane. A plane containing any two points contains all of the points on the line determined by those two points. Since the two points A and P on line l are on the plane, then all of the points of l are on the plane. Since the two points B and P on line m are on the plane, then all of the points of m are on the plane. Therefore, there is exactly one plane that contains the given intersecting lines. ◻

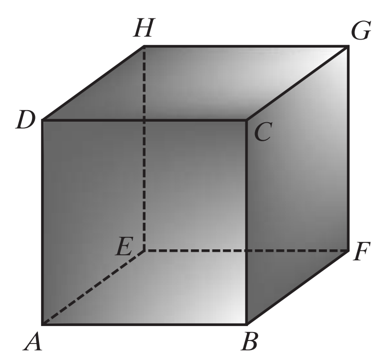
The definition of parallel lines gives us another set of points that must lie in a plane.

DEFINITION

Parallel lines in space are lines in the same plane that have no points in common.

This definition can be written as a biconditional: Two lines are parallel if and only if they are coplanar and have no points in common. If $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel lines in space, then they determine a plane if and only if they are two distinct lines.

We have seen that intersecting lines lie in a plane and parallel lines lie in a plane. For example, in the diagram, $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ so $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ lie in a plane. Also, $\overleftrightarrow{AB}$ and $\overleftrightarrow{BF}$ are intersecting lines so they lie in a plane. But there are some pairs of lines that do not intersect and are not parallel. In the diagram, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CG}$ are neither intersecting nor parallel lines. $\overleftrightarrow{AB}$ and $\overleftrightarrow{CG}$ are called *skew lines*. $\overleftrightarrow{BF}$ and $\overleftrightarrow{EH}$ are another pair of skew lines.



DEFINITION

Skew lines are lines in space that are neither parallel nor intersecting.

EXAMPLE 1

Does a triangle determine a plane?

Solution A triangle consists of three line segments and the three non-collinear points that are the endpoints of each pair of segments. Three non-collinear points determine a plane. That plane contains all of the points of the lines determined by the points. Therefore, a triangle determines a plane.

Exercises

Writing About Mathematics

1. Joel said that another definition for skew lines could be two lines that do not lie in the same plane. Do you agree with Joel? Explain why or why not.

2. Angelina said that if $\overline{AC}$ and $\overline{BD}$ intersect at a point, then A, B, C, and D lie in a plane and form a quadrilateral. Do you agree with Angelina? Explain why or why not.

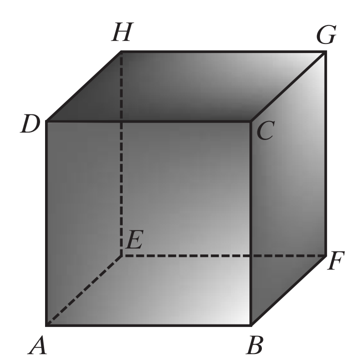
Developing Skills

3. $\overleftrightarrow{AB}$ is parallel to $\overleftrightarrow{CD}$ and $AB \neq CD$. Prove that A, B, C, and D must lie in a plane and form a trapezoid.

4. $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overleftrightarrow{AD} \parallel \overleftrightarrow{BC}$. Prove that A, B, C, and D must lie in a plane and form a parallelogram.

5. $\overline{AEC} \cong \overline{BED}$ and each segment is the perpendicular bisector of the other. Prove that A, B, C, and D must lie in a plane and form a square.

In 6–9, use the diagram at the right.



6. Name two pairs of intersecting lines.

7. Name two pairs of skew lines.

8. Name two pairs of parallel lines.

9. Which pairs of lines that you named in exercises 6, 7, and 8 are not lines in the same plane?

10. Let p represent "Two lines are parallel."

Let q represent "Two lines are coplanar."

Let r represent "Two lines have no point in common."

a. Write the biconditional "Two lines are parallel if and only if they are coplanar and have no points in common" in terms of p, q, r, and logical symbols.

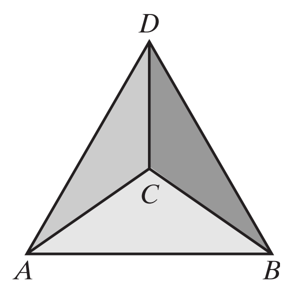
b. The biconditional is true. Show that q is true when p is true.

Applying Skills

11. A photographer wants to have a steady base for his camera. Should he choose a base with four legs or with three? Explain your answer.

12. Ken is building a tool shed in his backyard. He begins by driving four stakes into the ground to be the corners of a rectangular floor. He stretches strings from two of the stakes to the opposite stakes and adjusts the height of the stakes until the strings intersect. Explain how the strings assure him that the four stakes will all be in the same plane.

13. Each of four equilateral triangles has a common side with each of the three other triangles and form a solid called a *tetrahedron*. Prove that the triangles are congruent.

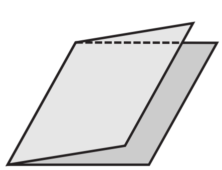


11-2 PERPENDICULAR LINES AND PLANES

Look at the floor, walls, and ceiling of the classroom. Each of these surfaces can be represented by a plane. Many of these planes intersect. For example, each wall intersects the ceiling and each wall intersects the floor. Each intersection can be represented by a line segment. This observation allows us to state the following postulate.

Postulate 11.3 If two planes intersect, then they intersect in exactly one line.

The Angle Formed by Two Intersecting Planes



Fold a piece of paper. The part of the paper on one side of the crease represents a half-plane and the crease represents the edge of the half-plane. The folded paper forms a dihedral angle.

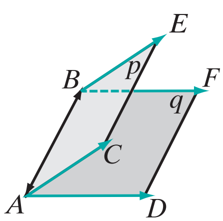
DEFINITION

A **dihedral angle** is the union of two half-planes with a common edge.

Each half-plane of a dihedral angle can be compared to a ray of an angle in a plane (or a **plane angle**) and the edge to the vertex of a plane angle. If we choose some point on the edge of a dihedral angle and draw, from this point, a ray in each half-plane perpendicular to the edge, we have drawn a plane angle.

DEFINITION

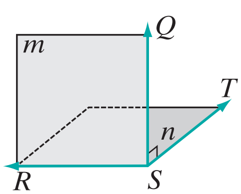
The **measure of a dihedral angle** is the measure of the plane angle formed by two rays each in a different half-plane of the angle and each perpendicular to the common edge at the same point of the edge.



A plane angle whose measure is the same as that of the dihedral angle can be drawn at any points on the edge of the dihedral angle. Each plane angle of a dihedral angle has the same measure. In the figure, planes p and q intersect at $\overleftrightarrow{AB}$. In plane p, $\overrightarrow{AC} \perp \overleftrightarrow{AB}$ and in plane q, $\overrightarrow{AD} \perp \overleftrightarrow{AB}$. The measure of the dihedral angle is equal to the measure of $\angle CAD$. Also, in plane p, $\overrightarrow{BE} \perp \overleftrightarrow{AB}$ and in plane q, $\overrightarrow{BF} \perp \overleftrightarrow{AB}$. The measure of the dihedral angle is equal to the measure of $\angle EBF$.

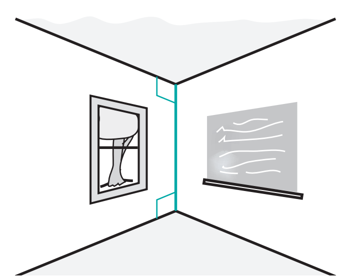
DEFINITION

Perpendicular planes are two planes that intersect to form a right dihedral angle.



In the diagram, $\angle QST$ is a right angle. In plane m, $\overrightarrow{SQ} \perp \overrightarrow{SR}$ and in plane n, $\overrightarrow{ST} \perp \overrightarrow{SR}$. The dihedral angle formed by half-planes of planes m and n with edge $\overleftrightarrow{RS}$ has the same measure as $\angle QST$. Therefore, the dihedral angle is a right angle, and $m \perp n$.

The floor and a wall of a room usually form a right dihedral angle. Look at the line that is the intersection of two adjacent walls of the classroom. This line intersects the ceiling in one point and intersects the floor in one point. This observation suggests the following theorem.



Theorem 11.3

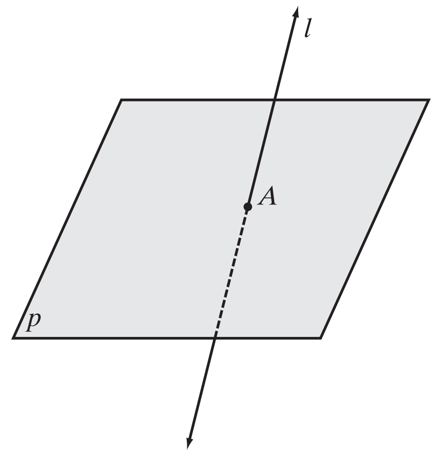
If a line not in a plane intersects the plane, then it intersects in exactly one point.

Given Line l is not in plane p and l intersects p.

Prove Line l intersects p in exactly one point.

Proof Use an indirect proof.

Assume that line l intersects the plane in two points. Then all of the points on line l lie in plane p, that is, the line lies in the plane. Because this contradicts the hypothesis that line l is not in plane p, the assumption must be false. A line not in a plane that intersects the plane, intersects it in exactly one point.



Again, look at the corner of the classroom in the figure on the previous page. The line that is the intersection of two adjacent walls intersects the ceiling so that the line is perpendicular to any line in the ceiling through the point of intersection. We say that this line is perpendicular to the plane of the ceiling.

DEFINITION

A line is perpendicular to a plane if and only if it is perpendicular to each line in the plane through the intersection of the line and the plane.

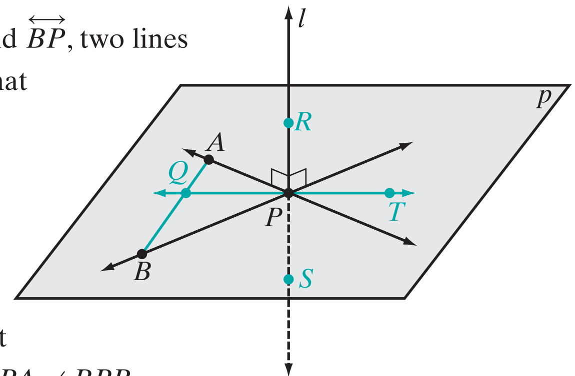
A **plane is perpendicular to a line** if the line is perpendicular to the plane.

It is easy to demonstrate that a line that is perpendicular to one line in a plane may not be perpendicular to the plane. For example, fold a rectangular sheet of paper. Draw a ray perpendicular to the crease with its endpoint on the crease. Keep the half of the folded sheet that does not contain the ray in contact with your desk. This is the plane. Move the other half to different positions. The ray that you drew is always perpendicular to the crease but is not always perpendicular to the plane. Based on this observation, we can state the following postulate.

Postulate 11.4 At a given point on a line, there are infinitely many lines perpendicular to the given line.

In order to prove that a line is perpendicular to a plane, the definition requires that we show that *every* line through the point of intersection is perpendicular to the given line. However, it is possible to prove that if line l is known to be perpendicular to each of two lines in plane p that intersect at point A, then l is perpendicular to plane p at A.

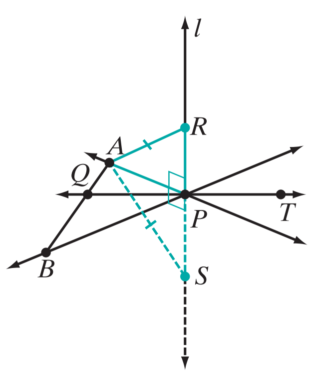
Theorem 11.4 If a line is perpendicular to each of two intersecting lines at their point of intersection, then the line is perpendicular to the plane determined by these lines.



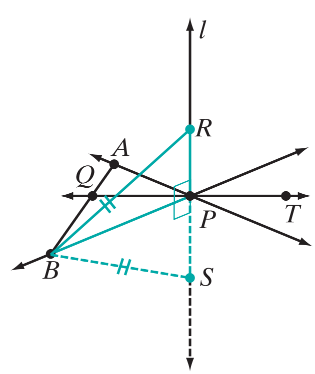
Given A plane p determined by $\overleftrightarrow{AP}$ and $\overleftrightarrow{BP}$, two lines that intersect at P. Line l such that $l \perp \overleftrightarrow{AP}$ and $l \perp \overleftrightarrow{BP}$.

Prove $l \perp p$

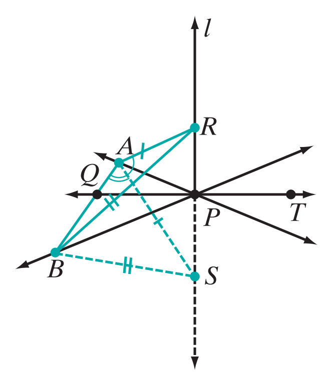
Proof To begin, let R and S be points on l such that P is the midpoint of $\overline{RS}$. Since it is given that $l \perp \overleftrightarrow{AP}$ and $l \perp \overleftrightarrow{BP}$, $\angle RPA$, $\angle SPA$, $\angle RPB$, and $\angle SPB$ are right angles and therefore congruent. Then let $\overleftrightarrow{PT}$ be any other line through P in plane p. Draw $\overleftrightarrow{AB}$ intersecting $\overleftrightarrow{PT}$ at Q. To prove this theorem, we need to show three different pairs of congruent triangles: $\triangle RPA \cong \triangle SPA$, $\triangle RPB \cong \triangle SPB$, and $\triangle RPQ \cong \triangle SPQ$. However, to establish the last congruence we must prove that $\triangle RAB \cong \triangle SAB$ and $\triangle RAQ \cong \triangle SAQ$.



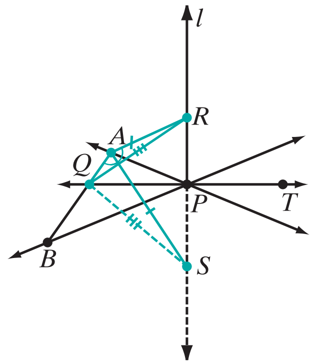
(1) $\triangle RPA \cong \triangle SPA$ by SAS and $\overline{AR} \cong \overline{AS}$.



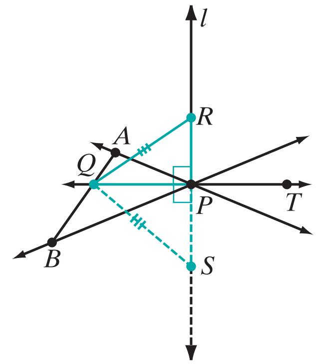
(2) $\triangle RPB \cong \triangle SPB$ by SAS and $\overline{BR} \cong \overline{BS}$.



(3) $\triangle RAB \cong \triangle SAB$ by SSS and $\angle RAB \cong \angle SAB$.



(4) $\triangle RAQ \cong \triangle SAQ$ by by SAS since $\angle RAB \cong \angle RAQ$ and $\angle SAB \cong \angle SAQ$, and $\overline{RQ} \cong \overline{SQ}$.



(5) $\triangle RPQ \cong \triangle SPQ$ by SSS and $\angle RPQ \cong \angle SPQ$.

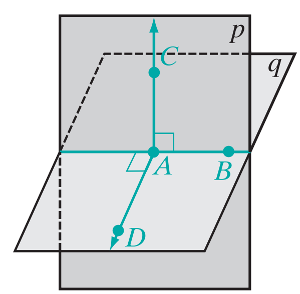
Now since $\angle RPQ$ and $\angle SPQ$ are a congruent linear pair of angles, they are right angles, and $l \perp \overleftrightarrow{PQ}$. Since $\overleftrightarrow{PQ}$, that is, $\overleftrightarrow{PT}$ can be *any* line in p through P, l is perpendicular to *every* line in plane p through point P. ■

Theorem 11.5a

If two planes are perpendicular to each other, one plane contains a line perpendicular to the other plane.

Given Plane $p \perp$ plane q

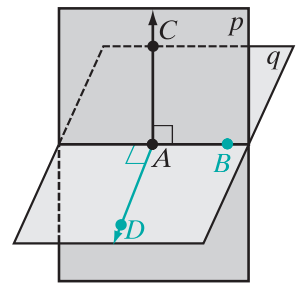
Prove A line in p is perpendicular to q and a line in q is perpendicular to p.



Proof If planes p and q are perpendicular to each other then they form a right dihedral angle. Let $\overleftrightarrow{AB}$ be the edge of the dihedral angle. In plane p, construct $\overleftrightarrow{AC} \perp \overleftrightarrow{AB}$, and in plane q, construct $\overleftrightarrow{AD} \perp \overleftrightarrow{AB}$. Since p and q from a right dihedral angle, $\angle CAD$ is a right angle and $\overleftrightarrow{AC} \perp \overleftrightarrow{AD}$. Two lines in plane p, $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$, are each perpendicular to $\overleftrightarrow{AD}$. Therefore, $\overleftrightarrow{AD} \perp p$. Similarly, two lines in plane q, $\overleftrightarrow{AB}$ and $\overleftrightarrow{AD}$, are each perpendicular to $\overleftrightarrow{AC}$. Therefore, $\overleftrightarrow{AC} \perp q$. ■

The converse of Theorem 11.5a is also true.

Theorem 11.5b If a plane contains a line perpendicular to another plane, then the planes are perpendicular.



Given $\overleftrightarrow{AC}$ in plane p and $\overleftrightarrow{AC} \perp q$

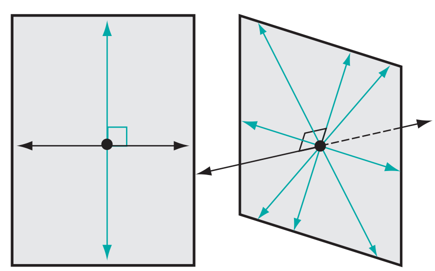
Prove $p \perp q$

Proof Let $\overleftrightarrow{AB}$ be the line of intersection of planes p and q. In plane q, draw $\overleftrightarrow{AD} \perp \overleftrightarrow{AB}$. Since $\overleftrightarrow{AC}$ is perpendicular to q, it is perpendicular to any line through A in q. Therefore, $\overleftrightarrow{AC} \perp \overleftrightarrow{AD}$ and also $\overleftrightarrow{AC} \perp \overleftrightarrow{AB}$. Thus, $\angle CAD$ is the plane angle of the dihedral angle formed by planes p and q. Since $\overleftrightarrow{AC}$ is perpendicular to $\overleftrightarrow{AD}$, $\angle CAD$ is a right angle. Therefore, the dihedral angle is a right angle, and p and q are perpendicular planes. ❑

This theorem and its converse can be stated as a biconditional.

Theorem 11.5 Two planes are perpendicular if and only if one plane contains a line perpendicular to the other.

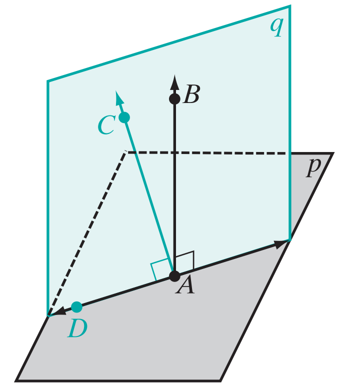
We know that in a plane, there is only one line perpendicular to a given line at a given point on the line. In space, there are infinitely many lines perpendicular to a given line at a given point on the line. These perpendicular lines are all in the same plane. However, only one line is perpendicular to a plane at a given point.



Theorem 11.6 Through a given point on a plane, there is only one line perpendicular to the given plane.

Given Plane p and $\overleftrightarrow{AB} \perp p$ at A.

Prove $\overleftrightarrow{AB}$ is the only line perpendicular to p at A.



Proof Use an indirect proof.

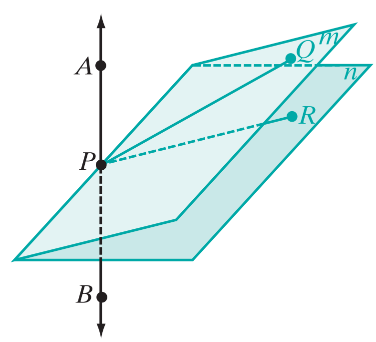
Assume that there exists another line, $\overleftrightarrow{AC} \perp p$ at A. Points A, B, and C determine a plane, q, that intersects plane p at $\overleftrightarrow{AD}$. Therefore, in plane q, $\overleftrightarrow{AB} \perp \overleftrightarrow{AD}$ and $\overleftrightarrow{AC} \perp \overleftrightarrow{AD}$. But in a given plane, there is only one line perpendicular to a given line at a given point. Our assumption is false, and there is only one line perpendicular to a given plane at a given point.

As we noted above, in space, there are infinitely many lines perpendicular to a given line at a given point. Any two of those intersecting lines determine a plane perpendicular to the given line. Each of these pairs of lines determine the same plane perpendicular to the given line.

Theorem 11.7 Through a given point on a line, there can be only one plane perpendicular to the given line.

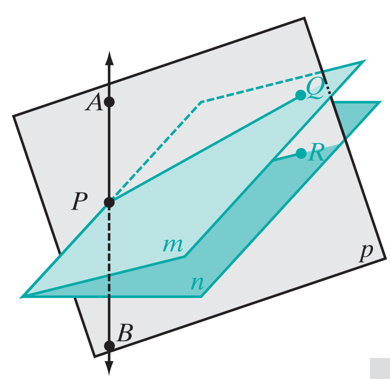
Given Any point P on $\overleftrightarrow{AB}$.

Prove There is only one plane perpendicular to $\overleftrightarrow{AB}$.



Proof Use an indirect proof.

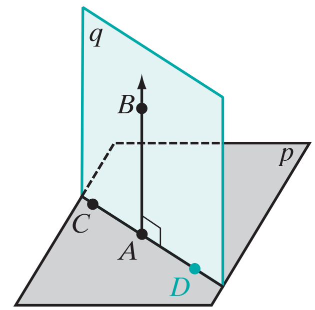
Assume that there are two planes, m and n, that are each perpendicular to $\overleftrightarrow{AB}$. Choose any point Q in m. Since $m \perp \overleftrightarrow{APB}$, $\overleftrightarrow{AP} \perp \overleftrightarrow{PQ}$. Points A, P, and Q determine a plane p that intersects plane n in a line $\overleftrightarrow{PR}$. Since $n \perp \overleftrightarrow{APB}$, $\overleftrightarrow{AP} \perp \overleftrightarrow{PR}$. Therefore, in plane p, $\overleftrightarrow{AP} \perp \overleftrightarrow{PQ}$ and $\overleftrightarrow{AP} \perp \overleftrightarrow{PR}$. But in a plane, at a given point there is one and only one line perpendicular to a given line. Our assumption must be false, and there is only one plane perpendicular to $\overleftrightarrow{AB}$ at P.



Theorem 11.8 If a line is perpendicular to a plane, then any line perpendicular to the given line at its point of intersection with the given plane is in the plane.

Given $\overleftrightarrow{AB} \perp$ plane p at A and $\overleftrightarrow{AB} \perp \overleftrightarrow{AC}$.

Prove $\overleftrightarrow{AC}$ is in plane p.

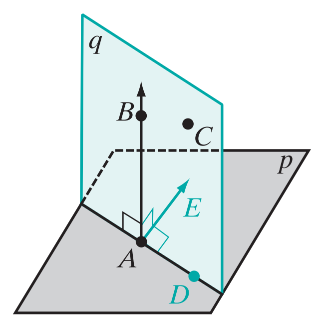


Proof Points A, B, and C determine a plane q. Plane q intersects plane p in a line, $\overleftrightarrow{AD}$. $\overleftrightarrow{AB} \perp \overleftrightarrow{AD}$ because $\overleftrightarrow{AB}$ is perpendicular to every line in p through A. It is given that $\overleftrightarrow{AB} \perp \overleftrightarrow{AC}$. Therefore, $\overleftrightarrow{AD}$ and $\overleftrightarrow{AC}$ in plane q are perpendicular to $\overleftrightarrow{AB}$ at A. But at a given point in a plane, only one line can be drawn perpendicular to a given line. Therefore, $\overleftrightarrow{AD}$ and $\overleftrightarrow{AC}$ are the same line, that is, C is on $\overleftrightarrow{AD}$. Since $\overleftrightarrow{AD}$ is the intersection of planes p and q, $\overleftrightarrow{AC}$ is in plane p. ◻

Theorem 11.9 If a line is perpendicular to a plane, then every plane containing the line is perpendicular to the given plane.

Given Plane p with $\overleftrightarrow{AB} \perp p$ at A, and C any point not on p.

Prove The plane q determined by A, B, and C is perpendicular to p.



Proof Let the intersection of p and q be $\overleftrightarrow{AD}$, so $\overleftrightarrow{AD}$ is the edge of the dihedral angle formed by p and q. Let $\overleftrightarrow{AE}$ be a line in p that is perpendicular to $\overleftrightarrow{AD}$. Since $\overleftrightarrow{AB} \perp p$, $\overleftrightarrow{AB}$ is perpendicular to every line in p through A. Therefore, $\overleftrightarrow{AB} \perp \overleftrightarrow{AD}$ and $\overleftrightarrow{AB} \perp \overleftrightarrow{AE}$. $\angle BAE$ is a plane angle whose measure is the measure of the dihedral angle. Since $\overleftrightarrow{AB} \perp \overleftrightarrow{AE}$, m$\angle BAE = 90$. Therefore, the dihedral angle is a right angle, and $q \perp p$. ◻

EXAMPLE 1

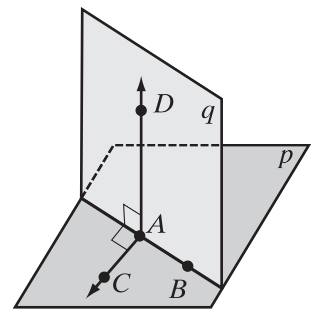
Show that the following statement is false.

Two planes perpendicular to the same plane have no points in common.

Solution Recall that a statement that is sometimes true and sometimes false is regarded to be false. Consider the adjacent walls of a room. Each wall is perpendicular to the floor but the walls intersect in a line. This counterexample shows that the given statement is false.

EXAMPLE 2

Planes p and q intersect in $\overleftrightarrow{AB}$. In p, $\overleftrightarrow{AC} \perp \overleftrightarrow{AB}$ and in q, $\overleftrightarrow{AD} \perp \overleftrightarrow{AB}$. If $m\angle CAD < 90$, is $p \perp q$?

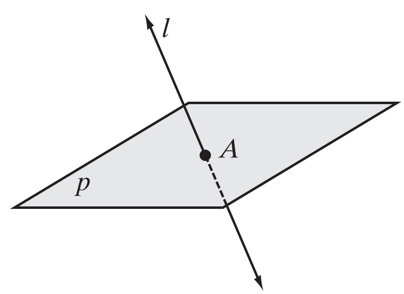


Solution Since in p, $\overleftrightarrow{AC} \perp \overleftrightarrow{AB}$, and in q, $\overleftrightarrow{AD} \perp \overleftrightarrow{AB}$, $\angle CAD$ is a plane angle whose measure is equal to the measure of the dihedral angle formed by the planes. Since $\angle CAD$ is not a right angle, then the dihedral angle is not a right angle, and the planes are not perpendicular.

EXAMPLE 3

Given: Line l intersects plane p at A, and l is not perpendicular to p.

Prove: There is at least one line through A in plane p that is not perpendicular to l.



Proof Use an indirect proof.

Let $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ be two lines through A in p. Assume that $l \perp \overleftrightarrow{AB}$ and $l \perp \overleftrightarrow{AC}$. Therefore, $l \perp p$ because if a line is perpendicular to each of two lines at their point of intersection, then the line is perpendicular to the plane determined by these lines. But it is given that l is not perpendicular to p. Therefore, our assumption is false, and l is not perpendicular to at least one of the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$.

Exercises

Writing About Mathematics

1. Carmen said if two planes intersect to form four dihedral angles that have equal measures, then the planes are perpendicular to each other. Do you agree with Carmen? Explain why or why not.

2. Each of three lines is perpendicular to the plane determined by the other two.

   a. Is each line perpendicular to each of the other two lines? Justify your answer.

   b. Name a physical object that justifies your answer.

Developing Skills

In 3–11, state whether each of the statements is true or false. If it is true, state a postulate or theorem that supports your answer. If it is false, describe or draw a counterexample.

3. At a given point on a given line, only one line is perpendicular to the line.

4. If A is a point in plane p and B is a point not in p, then no other point on $\overleftrightarrow{AB}$ is in plane p.

5. A line perpendicular to a plane is perpendicular to every line in the plane.

6. A line and a plane perpendicular to the same line at two different points have no points in common.

7. Two intersecting planes that are each perpendicular to a third plane are perpendicular to each other.

8. If $\overleftrightarrow{AB}$ is perpendicular to plane p at A and $\overleftrightarrow{AB}$ is in plane q, then $p \perp q$.

9. At a given point on a given plane, only one plane is perpendicular to the given plane.

10. If a plane is perpendicular to one of two intersecting lines, it is perpendicular to the other.

11. If a line is perpendicular to one of two intersecting planes, it is perpendicular to the other.

Applying Skills

12. Prove step 1 of Theorem 11.4.

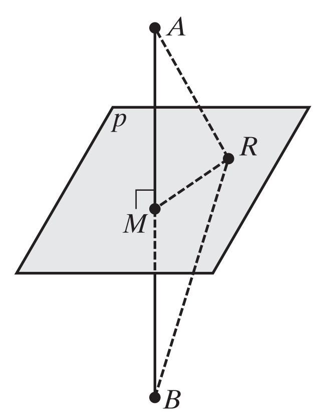
13. Prove step 3 of Theorem 11.4.

14. Prove step 5 of Theorem 11.4.

15. Prove that if a line segment is perpendicular to a plane at the midpoint of the line segment, then every point in the plane is equidistant from the endpoints of the line segment.

    Given: $\overline{AB} \perp$ plane p at M, the midpoint of $\overline{AB}$, and R is any point in plane p.

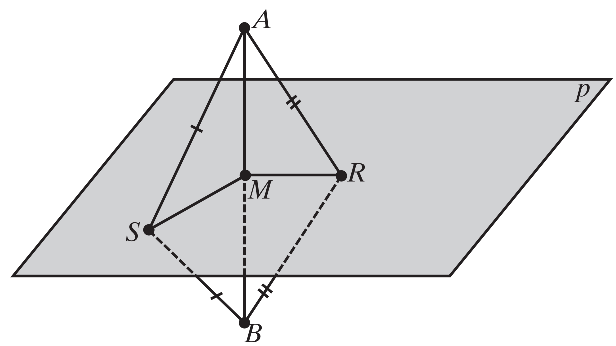
    Prove: $AR = BR$



16. Prove that if two points are each equidistant from the endpoints of a line segment, then the line segment is perpendicular to the plane determined by the two points and the midpoint of the line segment.

Given: M is the midpoint of $\overline{AB}$, $\overline{RA} \cong \overline{RB}$, and $\overline{SA} \cong \overline{SB}$.

Prove: $\overline{AB}$ is perpendicular to the plane determined by M, R, and S.



17. Equilateral triangle ABC is in plane p and $\overleftrightarrow{AD}$ is perpendicular to plane p. Prove that $\overline{BD} \cong \overline{CD}$.

18. $\overleftrightarrow{AB}$ and $\overleftrightarrow{AC}$ intersect at A and determine plane p. $\overleftrightarrow{AD}$ is perpendicular to plane p at A. If $AB = AC$, prove that $\triangle ABD \cong \triangle ACD$.

19. Triangle QRS is in plane p, $\overline{ST}$ is perpendicular to plane p, and $\angle QTS \cong \angle RTS$. Prove that $\overline{TQ} \cong \overline{TR}$.

20. Workers who are installing a new telephone pole position the pole so that it is perpendicular to the ground along two different lines. Prove that this is sufficient to prove that the telephone pole is perpendicular to the ground.

21. A telephone pole is perpendicular to the level ground. Prove that two wires of equal length attached to the pole at the same point and fastened to the ground are at equal distances from the pole.

11-3 PARALLEL LINES AND PLANES

Look at the floor, walls, and ceiling of the classroom. Each of these surfaces can be represented by a plane. Some of these surfaces, such as the floor and the ceiling, do not intersect. These can be represented as portions of *parallel planes*.

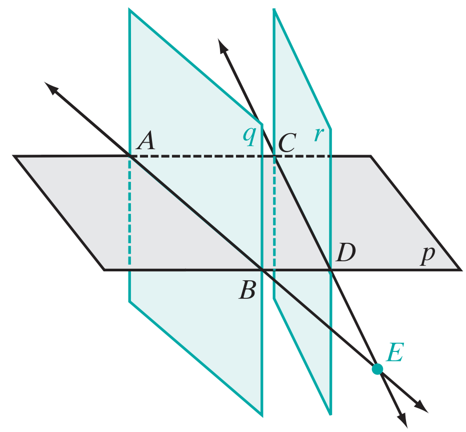
DEFINITION

Parallel planes are planes that have no points in common.

A **line is parallel to a plane** if it has no points in common with the plane.

EXAMPLE 1

Plane p intersects plane q in $\overleftrightarrow{AB}$ and plane r in $\overleftrightarrow{CD}$. Prove that if $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect, then planes q and r are not parallel.

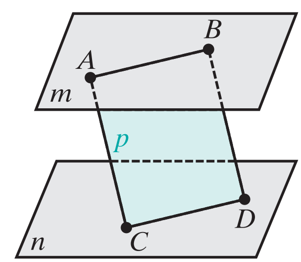


Proof Let E be the point at which $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect. Then E is a point on q and E is a point on r. Therefore, planes q and r intersect in at least one point and are not parallel.

Theorem 11.10

> If a plane intersects two parallel planes, then the intersection is two parallel lines.

Given Plane p intersects plane m at $\overleftrightarrow{AB}$ and plane n at $\overleftrightarrow{CD}$, $m \parallel n$.



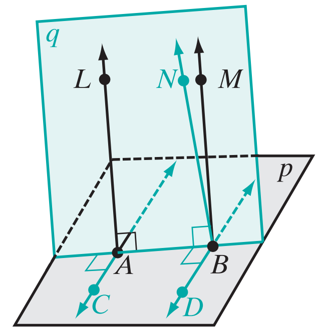
Prove $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$

Proof Use an indirect proof.

Lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are two lines of plane p. Two lines in the same plane either intersect or are parallel. If $\overleftrightarrow{AB}$ is not parallel to $\overleftrightarrow{CD}$ then they intersect in some point E. Since E is a point of $\overleftrightarrow{AB}$, then it is a point of plane m. Since E is a point of $\overleftrightarrow{CD}$, then it is a point of plane n. But $m \parallel n$ and have no points in common. Therefore, $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are two lines in the same plane that do not intersect, and $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$.

In a plane, two lines perpendicular to a given line are parallel. Can we prove that two lines perpendicular to a given plane are also parallel?

Theorem 11.11 Two lines perpendicular to the same plane are parallel.



Given Plane p, line $\overleftrightarrow{LA} \perp p$ at A, and line $\overleftrightarrow{MB} \perp p$ at B.

Prove $\overleftrightarrow{LA} \parallel \overleftrightarrow{MB}$

Proof We will construct line $\overleftrightarrow{NB}$ at B that is parallel to $\overleftrightarrow{LA}$, and show that $\overleftrightarrow{MB}$ and $\overleftrightarrow{NB}$ are the same line.

(1) Since it is given that $\overleftrightarrow{LA} \perp p$ at A, $\overleftrightarrow{LA}$ is perpendicular to any line in p through A, so $\overleftrightarrow{LA} \perp \overleftrightarrow{AB}$. Let q be the plane determined by $\overleftrightarrow{LA}$ and $\overleftrightarrow{AB}$. In plane p, draw $\overleftrightarrow{AC} \perp \overleftrightarrow{AB}$. Then $\angle LAC$ is a right angle, and p and q form a right dihedral angle.

(2) At point B, there exists a line $\overleftrightarrow{NB}$ in q that is parallel to $\overleftrightarrow{LA}$. If one of two parallel lines is perpendicular to a third line, then the other is perpendicular to the third line, that is, since $\overleftrightarrow{LA} \perp \overleftrightarrow{AB}$, then $\overleftrightarrow{NB} \perp \overleftrightarrow{AB}$.

(3) Draw $\overleftrightarrow{BD} \perp \overleftrightarrow{AB}$ in p. Because p and q form a right dihedral angle, $\angle NBD$ is a right angle, and so $\overleftrightarrow{NB} \perp \overleftrightarrow{BD}$.

(4) Therefore, $\overleftrightarrow{NB}$ is perpendicular to two lines in p at B (steps 2 and 3), so $\overleftrightarrow{NB}$ is perpendicular to p at B.

(5) But it is given that $\overleftrightarrow{MB} \perp p$ at B and there is only one line perpendicular to a given plane at a given point. Therefore, $\overleftrightarrow{MB}$ and $\overleftrightarrow{NB}$ are the same line, and $\overleftrightarrow{LA} \parallel \overleftrightarrow{MB}$. ❑

We have shown that two lines perpendicular to the same plane are parallel. Since parallel lines lie in the same plane, we have just proved the following corollary to this theorem:

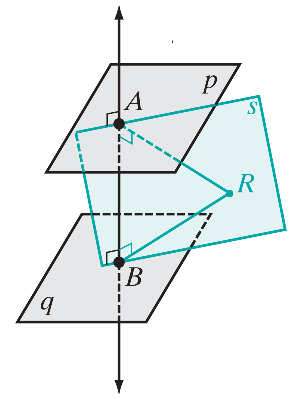
Corollary 11.11a Two lines perpendicular to the same plane are coplanar.

Theorem 11.12a If two planes are perpendicular to the same line, then they are parallel.

Given Plane $p \perp \overleftrightarrow{AB}$ at A and $q \perp \overleftrightarrow{AB}$ at B.

Prove $p \parallel q$

Proof Use an indirect proof.

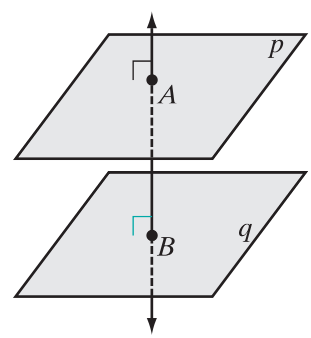


Assume that p is not parallel to q. Then p and q intersect in a line. Let R be any point on the line of intersection. Then A, B, and R determine a plane, s. In plane s, $\overleftrightarrow{AR} \perp \overleftrightarrow{AB}$ and $\overleftrightarrow{BR} \perp \overleftrightarrow{AB}$. But two lines in a plane that are perpendicular to the same line are parallel. Therefore, our assumption must be false, and $p \parallel q$.

Theorem 11.12b If two planes are parallel, then a line perpendicular to one of the planes is perpendicular to the other.

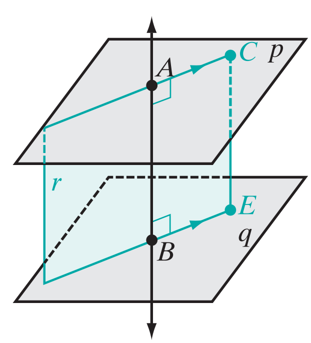
Given Plane p parallel to plane q, and $\overleftrightarrow{AB} \perp$ plane p and intersecting plane q at B

Prove $\overleftrightarrow{AB} \perp$ plane q

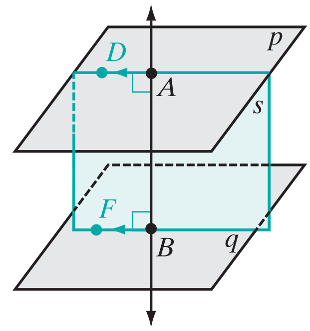


Proof To prove this theorem, we will construct two lines $\overleftrightarrow{BE}$ and $\overleftrightarrow{FB}$ in q that are both perpendicular to $\overleftrightarrow{AB}$. From this, we will conclude that $\overleftrightarrow{AB}$ is perpendicular to q.

(1) Let C be a point in p. Let r be the plane determined by A, B, and C intersecting q at $\overleftrightarrow{BE}$. Since p and q are parallel, $\overleftrightarrow{AC} \parallel \overleftrightarrow{BE}$. It is given that $\overleftrightarrow{AB} \perp$ plane p. Therefore, $\overleftrightarrow{AB} \perp \overleftrightarrow{AC}$. Then, in plane r, $\overleftrightarrow{AB} \perp \overleftrightarrow{BE}$.



(2) Let D be a point in p. Let s be the plane determined by A, B, and D intersecting q at $\overleftrightarrow{BF}$. Since p and q are parallel, $\overleftrightarrow{AD} \parallel \overleftrightarrow{BF}$. It is given that $\overleftrightarrow{AB} \perp$ plane p. Therefore, $\overleftrightarrow{AB} \perp \overleftrightarrow{AD}$. Then, in plane s, $\overleftrightarrow{AB} \perp \overleftrightarrow{BF}$.



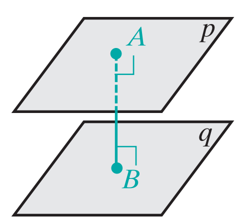
(3) If a line is perpendicular to each of two intersecting lines at their point of intersection, then the line is perpendicular to the plane determined by these lines. Therefore, $\overleftrightarrow{AB} \perp$ plane q.

Theorem 11.12a and 11.2b are converse statements. Therefore, we may write these two theorems as a biconditional.

Theorem 11.12

Two planes are perpendicular to the same line if and only if the planes are parallel.

Let p and q be two parallel planes. From A in p, draw $\overline{AB} \perp q$ at B. Therefore, $\overline{AB} \perp p$ at A. The distance from p to q is AB.



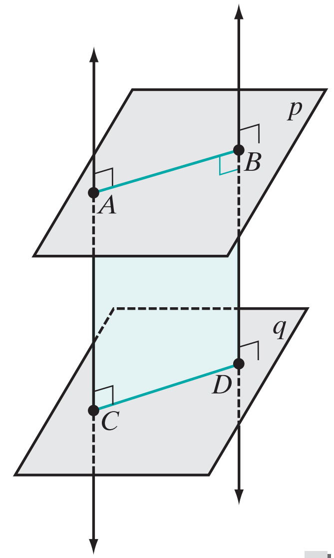
DEFINITION

The **distance between two planes** is the length of the line segment perpendicular to both planes with an endpoint on each plane.

Theorem 11.13 Parallel planes are everywhere equidistant.

Given Parallel planes p and q, with $\overline{AC}$ and $\overline{BD}$ each perpendicular to p and q with an endpoint on each plane.

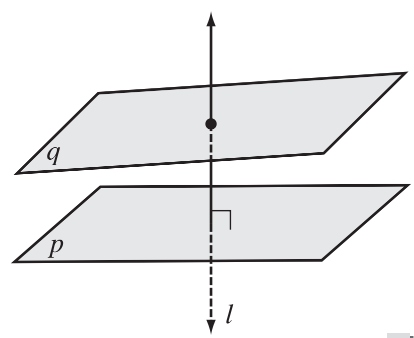
Prove $AC = BD$



Proof Two lines perpendicular to the same plane are both parallel and coplanar. Therefore, $\overline{AC} \parallel \overline{BD}$ and lie on the same plane. That plane intersects parallel planes p and q in parallel lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$. In the plane of $\overleftrightarrow{AC}$ and $\overleftrightarrow{BD}$, $ABDC$ is a parallelogram with a right angle, that is, a rectangle. Therefore, $\overline{AC}$ and $\overline{BD}$ are congruent and $AC = BD$.

EXAMPLE 2

Line l is perpendicular to plane p and line l is not perpendicular to plane q. Is $p \parallel q$?

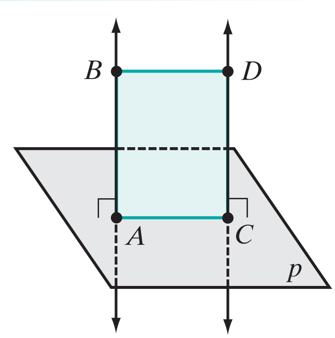


Solution Assume that $p \parallel q$. If two planes are parallel, then a line perpendicular to one is perpendicular to the other. Therefore, since l is perpendicular to plane p, l must be perpendicular to plane q. This contradicts the given statement that l is not perpendicular to q, and the assumption is false. Therefore, p is not parallel to q.

EXAMPLE 3

Given: $\overleftrightarrow{AB} \perp$ plane p at A, $\overleftrightarrow{CD} \perp$ plane p at C, and $AB = CD$.

Prove: A, B, C, and D are the vertices of a parallelogram.



Proof Two lines perpendicular to the same plane are parallel and coplanar. Therefore, since it is given that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are each perpendicular to p, they are parallel and coplanar. Since $AB = CD$ and segments of equal length are congruent, $ABCD$ is a quadrilateral with one pair of sides congruent and parallel. Therefore, $ABCD$ is a parallelogram. ■

Exercises

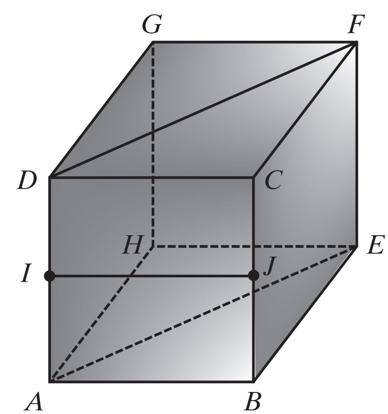
Writing About Mathematics

1. Two planes are perpendicular to the same plane. Are the planes parallel? Justify your answer.

2. Two planes are parallel to the same plane. Are the planes parallel? Justify your answer.

Developing Skills

In 3–9, each of the given statements is sometimes true and sometimes false. **a.** Give an example from the diagram to show that the statement can be true. **b.** Give a counterexample from the diagram to show that the statement can be false. In the diagram, each quadrilateral is a rectangle.



3. If two planes are perpendicular, a line parallel to one plane is perpendicular to the other.

4. Two planes parallel to the same line are parallel to each other.

5. Two lines perpendicular to the same line are parallel.

6. Two lines that do not intersect are parallel.

7. Two planes perpendicular to the same plane are parallel to each other.

8. Two lines parallel to the same plane are parallel.

9. If two lines are parallel, then a line that is skew to one line is skew to the other.

Applying Skills

10. ABC is an isosceles triangle with base $\overline{BC}$ in plane p. Plane $q \parallel p$ through point D on $\overline{AB}$ and point E on $\overline{AC}$. Prove that $\triangle ADE$ is isosceles.

11. Plane p is perpendicular to $\overleftrightarrow{PQ}$ at Q and two points in p, A and B, are equidistant from P. Prove that $\overline{AQ} \cong \overline{BQ}$.

12. Noah is building a tool shed. He has a rectangular floor in place and wants to be sure that the posts that he erects at each corner of the floor as the ends of the walls are parallel. He erects each post perpendicular to the floor. Are the posts parallel to each other? Justify your answer.

13. Noah wants the flat ceiling on his tool shed to be parallel to the floor. Two of the posts are 80 inches long and two are 78 inches long. Will the ceiling be parallel to the floor? Justify your answer. What must Noah do to make the ceiling parallel to the floor?

11-4 SURFACE AREA OF A PRISM

Polyhedron

In the plane, a polygon is a closed figure that is the union of line segments. In space, a *polyhedron* is a figure that is the union of polygons.

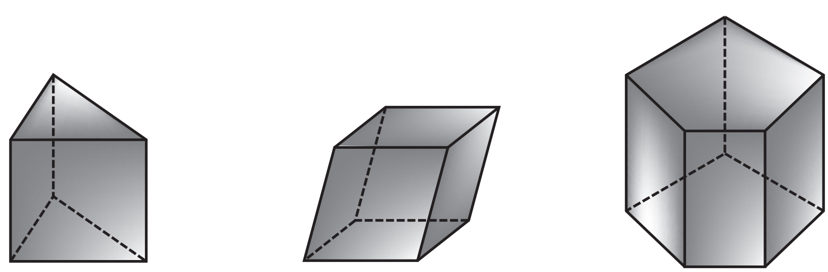
DEFINITION

A **polyhedron** is a three-dimensional figure formed by the union of the surfaces enclosed by plane figures.

The portions of the planes enclosed by a plane figure are called the **faces** of the polyhedron. The intersections of the faces are the **edges** of the polyhedron and the intersections of the edges are the **vertices** of the polyhedron.

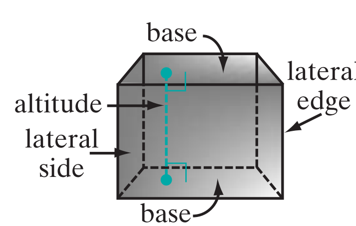
DEFINITION

A **prism** is a polyhedron in which two of the faces, called the **bases** of the prism, are congruent polygons in parallel planes.



Examples of prisms

The surfaces between corresponding sides of the bases are called the **lateral sides** of the prism and the common edges of the lateral sides are called the **lateral edges**. An **altitude** of a prism is a line segment perpendicular to each of the bases with an endpoint on each base. The **height** of a prism is the length of an altitude.

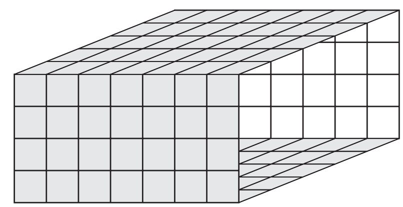


Since the bases are parallel, the corresponding sides of the bases are congruent, parallel line segments. Therefore, each lateral side has a pair of congruent, parallel sides, the corresponding edges of the bases, and are thus parallelograms. The other pair of sides of these parallelograms, the lateral edges, are also congruent and parallel. Therefore, we can make the following statement:

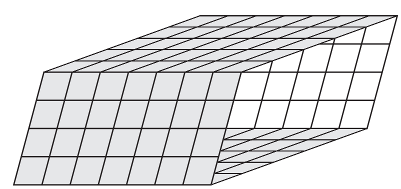
► **The lateral edges of a prism are congruent and parallel.**

DEFINITION

A **right prism** is a prism in which the lateral sides are all perpendicular to the bases. All of the lateral sides of a right prism are rectangles.



Using graph paper, cut two 7-by-5 rectangles and two 7-by-4 rectangles. Use tape to join the 7-by-4 rectangles to opposite sides of one of the 7-by-5 rectangles along the congruent edges. Then join the other 7-by-5 rectangle to the congruent edges forming four of the six sides of a prism. Place the prism on your desk on its side so that one pair of congruent rectangles are the bases and the lateral edges are perpendicular to the bases. Are the opposite faces parallel? What would be the shape and size of the two missing sides? Then move the top base so that the lateral edges are not perpendicular to the bases. The figure is no longer a right prism. Are the opposite faces parallel? What would be the shape and size of the two missing sides?

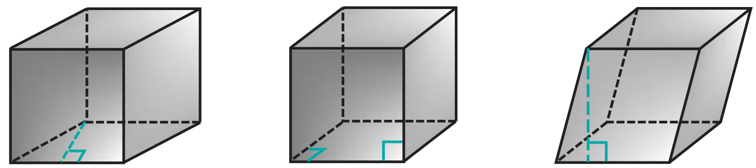


Cut two more 7-by-5 rectangles and two parallelograms that are not rectangles. Let the lengths of two of the sides of the parallelograms be 7 and the length of the altitude to these sides be 4. Join the parallelograms to opposite sides of one of the rectangles along congruent sides. Then join the other rectangle to congruent edges forming four of the six sides of a prism. Place the prism on your desk so that the rectangles are the bases and the lateral edges are perpendicular to the bases. Are the opposite faces parallel? Is the prism a right prism? What would be the shape and size of the two missing lateral sides? Move the top base of the prism so that the sides are not perpendicular to the bases. Are the opposite faces parallel? What would be the shape and size of the two missing sides? Now turn this prism so that the parallelograms are the bases and the edges of the rectangles are perpendicular to the bases. What is the shape of the two missing sides? Is this a right prism?

The solids that you have made are called *parallelepipeds*.

DEFINITION

A **parallelepiped** is a prism that has parallelograms as bases.

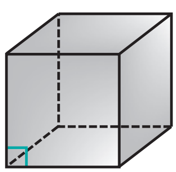


Examples of parallelepipeds

Rectangular Solids

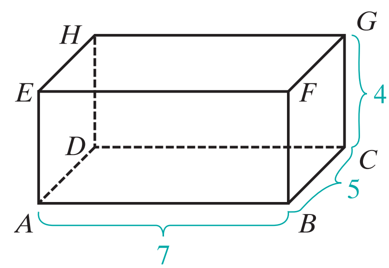
DEFINITION

A **rectangular parallelepiped** is a parallelepiped that has rectangular bases and lateral edges perpendicular to the bases.



A rectangular parallelepiped is usually called a **rectangular solid**. It is the most common prism and is the union of six rectangles. Any two parallel rectangles of a rectangular solid can be the bases.

In the figure, $ABCDEFGH$ is a rectangular solid. The bases $ABCD$ and $EFGH$ are congruent rectangles with $AB = EF = CD = GH = 7$ and $BC = FG = DA = HE = 5$. Two of the lateral sides are rectangles $ABFE$ and $DCGH$ whose dimensions are 7 by 4. The other two lateral sides are $BCGF$ and $ADHE$ whose dimensions are 5 by 4.



- The area of each base: $7 \times 5 = 35$
- The area of each of two lateral sides: $7 \times 4 = 28$
- The area of each of the other two lateral sides: $5 \times 4 = 20$

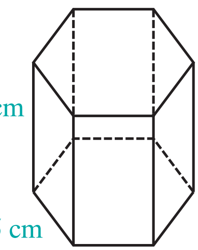
The **lateral area** of the prism is the sum of the areas of the lateral faces. The **total surface area** is the sum of the lateral area and the areas of the bases.

- The lateral area of the prism is $2(28) + 2(20) = 96$.
- The area of the bases are each $2(35) = 70$.
- The surface area of the prism is $96 + 70 = 166$.

EXAMPLE 1

The bases of a right prism are regular hexagons. The length of each side of a base is 5 centimeters and the height of the prism is 8 centimeters. Describe the number, shape, and size of the lateral sides.

Solution A hexagon has six sides. Because the base is a regular hexagon, it has six congruent sides, and therefore, the prism has six lateral sides. Since it is a right prism, the lateral sides are rectangles. The length of each of two edges of a lateral side is the length of an edge of a base, 5 centimeters. The length of each of the other two edges of a lateral side is the height of the prism, 8 centimeters.



Answer There are six lateral sides, each is a rectangle that is 8 centimeters by 5 centimeters.

EXAMPLE 2

The bases of a right prism are equilateral triangles. The length of one edge of a base is 4 inches and the height of the prism is 5 inches.

a. How many lateral sides does this prism have and what is their shape?

b. What is the lateral area of the prism?

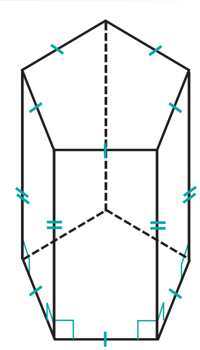
Solution **a.** Because this is a prism with a triangular base, the prism has three lateral sides. Because it is a right prism, the lateral sides are rectangles.

b. For each rectangular side, the length of one pair of edges is the length of an edge of the base, 4 inches. The height of the prism, 5 inches, is the length of a lateral edge. Therefore, the area of each lateral side is 4(5) or 20 square inches. The lateral area of the prism is $3(20) = 60$ square inches.

Answers **a.** 3 **b.** 60 square inches

EXAMPLE 3

The lateral sides of a prism are five congruent rectangles. Prove that the bases are equilateral pentagons.



Proof The five lateral sides are congruent rectangles. Two parallel sides of each rectangle are lateral edges. The other two parallel sides of each rectangle are edges of the bases. Each edge of a base is a side of a rectangle. Therefore, the base has five sides. The corresponding sides of the congruent rectangles are congruent. Therefore, the edges of each base are congruent. The bases are equilateral pentagons.

Exercises

Writing About Mathematics

1. Cut a 12-by-5 rectangle from graph paper, fold it into three 4-by-5 rectangles and fasten the two sides of length 5 with tape. Then cut a 16-by-5 rectangle from graph paper, fold it into four 4-by-5 rectangles and fasten the two sides of length 5 with tape. Let the open ends of each figure be the bases of a prism.

a. What is the shape of a base of the prism formed from the 12-by-5 paper? Can the shape of this base be changed? Explain your answer.

b. What is the shape of a base of the prism formed from the 16-by-5 paper? Can the shape of this base be changed? Explain your answer.

c. Are both figures always right prisms? Explain your answer.

2. A prism has bases that are rectangles, two lateral faces that are rectangles and two lateral faces that are parallelograms that are not rectangles.

a. Is an altitude of the solid congruent to an altitude of one of the rectangular faces? Explain your answer.

b. Is an altitude of the solid congruent to an altitude of one of the faces that are parallelograms? Explain your answer.

Developing Skills

In 3–6, find the surface area of each of the rectangular solid with the given dimensions.

3. 5.0 cm by 8.0 cm by 3.0 cm

4. 15 in. by 10.0 in. by 2.0 ft

5. 2.5 ft by 8.0 ft by 12 ft

6. 56.3 cm by 18.7 cm by 0.500 m

7. The bases of a prism are right triangles whose edges measure 9.00 centimeters, 40.0 centimeters, and 41.0 centimeters. The lateral sides of the prism are perpendicular to the bases. The height of the prism is 14.5 centimeters.

a. What is the shape of the lateral sides of the prism?

b. What are the dimensions of each lateral side of the prism?

c. What is the total surface area of the prism?

8. The bases of a right prism are isosceles triangles. The lengths of the sides of the bases are 5 centimeters, 5 centimeters, and 6 centimeters. The length of the altitude to the longest side of a base is 4 centimeters. The height of the prism is 12 centimeters.

a. What is the shape of the lateral sides of the prism?

b. What are the dimensions of each lateral side of the prism?

c. What is the total surface area of the prism?

9. How many faces does a parallelepiped have? Justify your answer.

10. A prism has bases that are trapezoids. Is the prism a parallelepiped? Justify your answer.

11. The length of an edge of a cube is 5.20 inches. What is the total surface area of the cube to the nearest square inch?

Applying Skills

12. The bases of a parallelepiped are $ABCD$ and $EFGH$, and $ABCD \cong EFGH$. Prove that $\overline{AE} \parallel \overline{BF} \parallel \overline{CG} \parallel \overline{DH}$ and that $AE = BF = CG = DH$.

13. The bases of a prism are $\triangle ABC$ and $\triangle DEF$, and $\triangle ABC \cong \triangle DEF$. The line through A perpendicular to the plane of $\triangle ABC$ intersects the plane of $\triangle DEF$ at D, and the line through B perpendicular to the plane of $\triangle ABC$ intersects the plane of $\triangle DEF$ at E.

a. Prove that the lateral faces of the prism are rectangles.

b. When are the lateral faces of the prism congruent polygons? Justify your answer.

14. A right prism has bases that are squares. The area of one base is 81 square feet. The lateral area of the prism is 144 square feet. What is the length of the altitude of the prism?

15. Show that the edges of a parallelepiped form three sets of parallel line segments.

16. The lateral faces of a parallelepiped are squares. What must be the shape of the bases? Justify your answer.

17. The lateral faces of a parallelepiped are squares. One angle of one of the bases is a right angle. Prove that the parallelepiped is a *cube*, that is, a rectangular parallelepiped with congruent faces.

18. The walls, floor, and ceiling of a room form a rectangular solid. The total surface area of the room is 992 square feet. The dimensions of the floor are 12 feet by 20 feet.

a. What is the lateral area of the room?

b. What is the height of the room?

Hands-On Activity

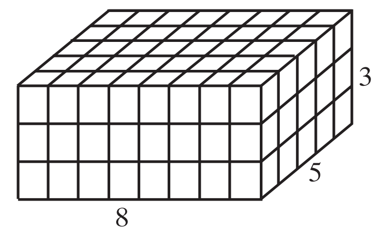
Let the bases of a prism be $ABCD$ and $A'B'C'D'$. Use the prisms that you made out of graph paper for page 441 to demonstrate each of the following.

1. When $\overline{AA'}$ is perpendicular to the planes of the bases, the lateral faces are rectangles and the height of the each lateral face is the height of the prism.

2. When a line through A perpendicular to the bases intersects the plane of $A'B'C'D'$ at a point on $\overleftrightarrow{A'B'}$, two of the lateral faces are rectangles and two are parallelograms. The height of the prism is the height of the lateral faces that are parallelograms but the height of the rectangles is not equal to the height of the prism.

3. When a line through A perpendicular to the bases intersects the plane of $A'B'C'D'$ at a point that is not on a side of $A'B'C'D'$, then the lateral faces are parallelograms and the height of the prism is not equal to the heights of the parallelogram.

11-5 VOLUME OF A PRISM

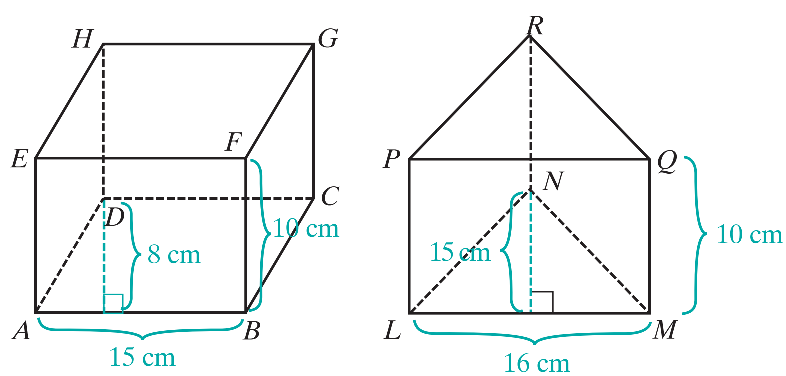


A cube whose edges each measure 1 centimeter is a unit of volume called a **cubic centimeter**. If the bases of a rectangular solid measure 8 centimeters by 5 centimeters, we know that the area of a base is 8×5 or 40 square centimeters and that 40 cubes each with a volume of 1 cubic centimeter can fill one base. If the height of the solid is 3 centimeters, we know that we can place 3 layers with 40 cubic centimeters in each layer to fill the rectangular solid. The volume of the solid is 40×3 or 120 cubic centimeters. The volume of the rectangular solid is the area of the base times the height. This can be applied to any prism and suggests the following postulate.

Postulate 11.5 The volume of a prism is equal to the area of the base times the height.

If V represents the volume of a prism, B represents the area of the base, and h the height of the prism, then:

$$V = Bh$$



The figure shows two prisms. One is a parallelepiped with parallelograms $ABCD$ and $EFGH$ as bases and rectangular faces $ABFE$, $BCGF$, $CDHG$, and $DAEH$. If AB is 15 centimeters and the length of the altitude from D to $\overline{AB}$ is 8 centimeters, then the area of the base $ABCD$ is 15×8 or 120 square centimeters. If BF, the height of the parallelepiped, is 10 centimeters, then:

$$\begin{aligned}\text{Volume of the parallelepiped} &= Bh \\ &= 120 \times 10 \\ &= 1{,}200 \text{ cubic centimeters}\end{aligned}$$

The other prism has bases that are triangles, $\triangle LMN$ and $\triangle PQR$. If LM is 16 centimeters and the length of the altitude to $\overline{LM}$ is 15 centimeters, then the area of a base is $\frac{1}{2}(16)(15)$ or 120 square centimeters. If the height of this prism is 10 centimeters, then:

$$\begin{aligned}\text{Volume of the triangular prism} &= Bh\\ &= 120 \times 10\\ &= 1{,}200 \text{ cubic centimeters}\end{aligned}$$

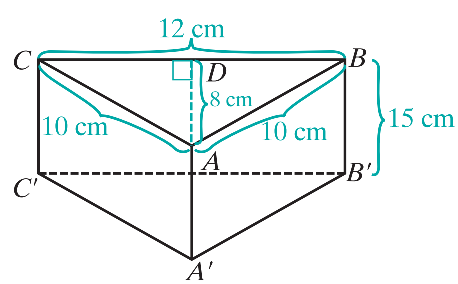
Note that for these two prisms, the areas of the bases are equal and the heights of the prisms are equal. Therefore, the volumes of the prisms are equal. This is true in general since volume is defined as the area of the base times the height of the prism.

The terms "base" and "height" are used in more than one way when describing a prism. For example, each of the congruent polygons in parallel planes is a base of the prism. The distance between the parallel planes is the height of the prism. In order to find the area of a base that is a triangle or a parallelogram, we use the length of a base and the height of the triangle or parallelogram. When finding the area of a lateral face that is a parallelogram, we use the length of the base and the height of that parallelogram. Care must be taken in distinguishing to what line segments the words "base" and "height" refer.

EXAMPLE 1

The bases of a right prism are $\triangle ABC$ and $\triangle A'B'C'$ with D a point on $\overline{CB}$, $\overline{AD} \perp \overline{BC}$, $AB = 10$ cm, $AC = 10$ cm, $BC = 12$ cm, $AD = 8$ cm, and $BB' = 15$ cm. Find the volume of the prism.

Solution Since this is a right prism, all of the lateral faces are rectangles and the height of the prism, AA', is the height of each face.



Each base is an isosceles triangle. The length of the base of the isosceles triangle is $BC = 12$ cm, and the length of the altitude to the base of the triangle is $AD = 8$ cm.

$$\begin{aligned}\text{Area of } \triangle ABC &= \tfrac{1}{2}(BC)(AD)\\ &= \tfrac{1}{2}(12)(8)\\ &= 48\end{aligned}$$

Since the prism is a right prism, the height of the prism is $BB' = 15$.

$$\begin{aligned}\text{Volume of the prism} &= (\text{area of a base})(\text{height of the prism})\\ &= (48)(15)\\ &= 720 \text{ cubic centimeters} \quad \textit{Answer}\end{aligned}$$

Exercises

Writing About Mathematics

1. Zoe said that if two solids have equal volumes and equal heights, then they must have congruent bases. Do you agree with Zoe? Justify your answer.

2. Piper said that the height of a prism is equal to the height of each of its lateral sides only if all of the lateral sides of the prism are rectangles. Do you agree with Piper? Explain why or why not.

Developing Skills

In 3–7, find the volume of each prism.

3. The area of the base is 48 square feet and the height is 18 inches.

4. The prism is a rectangular solid whose dimensions are 2.0 feet by 8.5 feet by 1.6 feet.

5. One base is a right triangle whose legs measure 5 inches and 7 inches. The height of the prism is 9 inches.

6. One base is a square whose sides measure 12 centimeters and the height of an altitude is 75 millimeters.

7. One base is parallelogram $ABCD$ and the other is parallelogram $A'B'C'D'$, $AB = 47$ cm, the length of the perpendicular from D to $\overline{AB} = 56$ cm, $\overline{AA'} \perp \overline{AB}$, $\overline{AA'} \perp \overline{AD}$, and $AA' = 19$ cm.

Applying Skills

8. A fish tank in the form of a rectangular solid is to accommodate 6 fish, and each fish needs at least 7,500 cubic centimeters of space. The dimensions of the base are to be 30 centimeters by 60 centimeters. What is the minimum height that the tank needs to be?

9. A parallelepiped and a rectangular solid have equal volume and equal height. The bases of the rectangular solid measure 15 centimeters by 24 centimeters. If the length of one side of a base of the parallelepiped measures 20 centimeters, what must be the length of the altitude to that base?

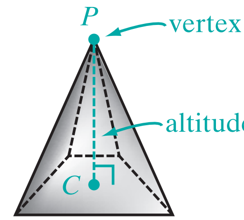
10. A prism whose bases are triangles and one whose bases are squares have equal volume and equal height. Triangle ABC is one base of the triangular prism and $PQRS$ is one base of the square prism. If $\overline{CD}$ is the altitude from C to $\overline{AB}$ and $AB = PQ$, what is the ratio of AB to CD? Justify your answer.

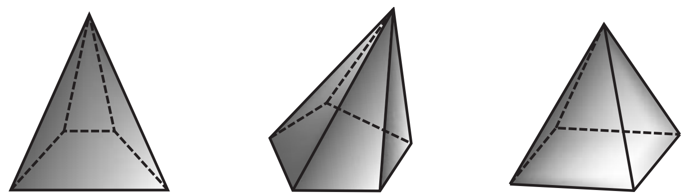
11. Prove that a plane that lies between the bases of a triangular prism and is parallel to the bases intersects the lateral sides of the prism in a triangle congruent to the bases.

12. Prove that the lateral area of a right prism is equal to the perimeter of a base times the height of the prism.

11-6 PYRAMIDS

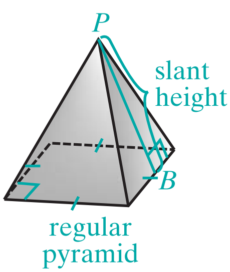
A **pyramid** is a solid figure with a base that is a polygon and lateral faces that are triangles. Each lateral face shares a common edge with the base and a common edge with two other lateral faces. All lateral edges meet in a point called the **vertex**. The **altitude** of a pyramid is the perpendicular line segment from the vertex to the base ($\overline{PC}$ in the diagram on the right.) The **height** of a pyramid is the length of the altitude.





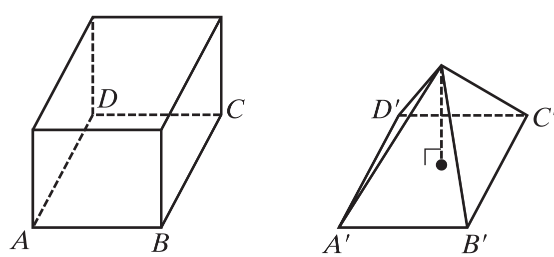
Examples of pyramids

Regular Pyramids



A **regular pyramid** is a pyramid whose base is a regular polygon and whose altitude is perpendicular to the base at its center. The lateral edges of a regular polygon are congruent. Therefore, the lateral faces of a regular pyramid are isosceles triangles. The length of the altitude of a triangular lateral face of a regular pyramid, PB, is the **slant height** of the pyramid.

Surface Area and Volume of a Pyramid



The figure shows a prism and a pyramid that have congruent bases and equal heights. If we were to fill the pyramid with water and empty the water into the prism, we would need to do this three times to fill the prism. Thus, since the volume of a prism is given by Bh:

$$\textbf{Volume of a pyramid} = \frac{1}{3}Bh$$

The lateral area of a pyramid is the sum of the areas of the faces. The total surface area is the lateral area plus the area of the bases.

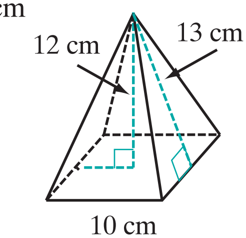
EXAMPLE 1

A regular pyramid has a square base and four lateral sides that are isosceles triangles. The length of an edge of the base is 10 centimeters and the height of the pyramid is 12 centimeters. The length of the altitude to the base of each lateral side is 13 centimeters.

a. What is the total surface area of the pyramid?

b. What is the volume of the pyramid?

Solution Let e be the length of a side of the square base: $e = 10$ cm



Let h_p be the height of the pyramid: $h_p = 12$ cm

Let h_s be the slant height of the pyramid: $h_s = 13$ cm

a. The base is a square with $e = 10$ cm.

The area of the base is $e^2 = (10)^2 = 100$ cm^2.

Each lateral side is an isosceles triangle. The length of each base, e, is 10 centimeters and the height, h_s, is 13 centimeters.

The area of each lateral side is $\frac{1}{2}eh_s = \frac{1}{2}(10)(13) = 65$ cm^2.

The total surface area of the pyramid is $100 + 4(65) = 360$ cm^2.

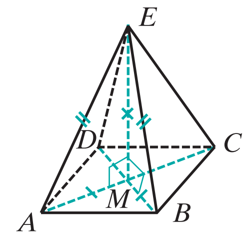
b. The volume of the prism is one-third of the area of the base times the height of the pyramid.

$$\begin{aligned} V &= \tfrac{1}{3}Bh_p \\ &= \tfrac{1}{3}(100)(12) \\ &= 400 \text{ cm}^3 \end{aligned}$$

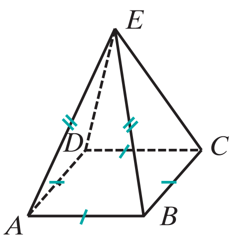
Answers **a.** 360 cm^2 **b.** 400 cm^3

Properties of Regular Pyramids

The base of a regular pyramid is a regular polygon and the altitude is perpendicular to the base at its center. The *center of a regular polygon* is defined as the point that is equidistant to its vertices. In a regular polygon with three sides, an equilateral triangle, we proved that the perpendicular bisector of the sides of the triangle meet in a point that is equidistant from the vertices of the triangle. In a regular polygon with four sides, a square, we know that the diagonals are congruent. Therefore, the point at which the diagonals bisect each other is equidistant from the vertices. In Chapter 13, we will show that for any regular polygon, a point equidistant from the vertices exists. For now, we can use this fact to show that the lateral sides of a regular pyramid are isosceles triangles.



For example, consider a regular pyramid with square $ABCD$ for a base and vertex E. The diagonals of $ABCD$ intersect at M and $AM = BM = CM = DM$. Since $\overline{EM}$ is perpendicular to the base, it is perpendicular to any line in the base through M. Therefore, $\overline{EM} \perp \overline{MA}$ and $\angle EMA$ is a right angle. Also, $\overline{EM} \perp \overline{MB}$ and $\angle EMB$ is a right angle. Since the diagonals of a square are congruent and bisect each other, $\overline{MA} \cong \overline{MB}$. Then since $\overline{EM} \cong \overline{EM}$, $\triangle EMA \cong \triangle EMB$ by SAS, and $\overline{EA} \cong \overline{EB}$ because they are corresponding parts of congruent triangles. Similar reasoning will lead us to conclude that $\overline{EB} \cong \overline{EC}$, $\overline{EC} \cong \overline{ED}$, and $\overline{ED} \cong \overline{EA}$. A similar proof can be given for any base that is a regular polygon. Therefore, we can make the following statement:



► **The lateral faces of a regular pyramid are isosceles triangles.**

In the regular pyramid with base $ABCD$ and vertex E,

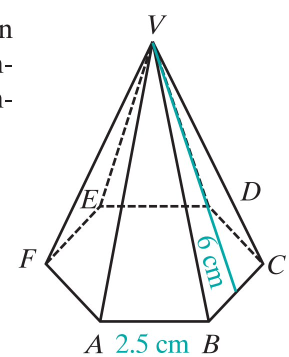
$$AB = BC = CD = DA \quad \text{and} \quad AE = BE = CE = DE$$

Therefore, $\triangle ABE \cong \triangle BCE \cong \triangle CDE \cong \triangle DAE$, that is, the lateral faces of the pyramid are congruent.

► **The lateral faces of a regular pyramid are congruent.**

EXAMPLE 2

A regular pyramid has a base that is the hexagon $ABCDEF$ and vertex at V. If the length $\overline{AB}$ is 2.5 centimeters, and the slant height of the pyramid is 6 centimeters, find the lateral area of the pyramid.



Solution The slant height of the pyramid is the height of a lateral face. Therefore:

$$\begin{aligned} \text{Area of } \triangle ABV &= \tfrac{1}{2}bh \\ &= \tfrac{1}{2}(2.5)(6) \\ &= \tfrac{15}{2} \text{ cm}^2 \end{aligned}$$

The lateral faces of the regular pyramid are congruent. Therefore, they have equal areas. There are six lateral faces.

$$\begin{aligned} \text{Lateral area of the pyramid} &= 6\left(\tfrac{15}{2}\right) \\ &= 45 \text{ cm}^2 \end{aligned}$$ *Answer*

Exercises

Writing About Mathematics

1. Martin said that if the base of a regular pyramid is an equilateral triangle, then the foot of the altitude of the pyramid is the point at which the altitudes of the base intersect. Sarah said that it is the point at which the medians intersect. Who is correct? Justify your answer.

2. Are the lateral faces of a pyramid always congruent triangles? Explain your answer.

Developing Skills

In 3–5, the information refers to a regular pyramid. Let e be the length of an edge of the base and h_s be the slant height. Find lateral area of each pyramid.

3. The pyramid has a square base; $e = 12$ cm, $h_s = 10$ cm

4. The pyramid has a triangular base; $e = 8.0$ ft, $h_s = 10$ ft

5. The pyramid has a base that is a hexagon; $e = 48$ cm, $h_s = 32$ cm

In 6–9, the information refers to a regular pyramid. Let e be the length of an edge of the base and h_p be the height of the pyramid. Find the volume of each pyramid.

6. The area of the base is 144 square centimeters; $h_p = 12$ cm

7. The area of the base is 27.6 square inches; $h_p = 5.0$ in.

8. The pyramid has a square base; $e = 2$ ft, $h_p = 1.5$ ft

9. The pyramid has a square base; $e = 22$ cm, $h_p = 14$ cm

10. The volume of a pyramid is 576 cubic inches and the height of the pyramid is 18 inches. Find the area of the base.

Applying Skills

11. A tetrahedron is a solid figure made up of four congruent equilateral triangles. Any one of the triangles can be considered to be the base and the other three to be the lateral sides of a regular pyramid. The length of a side of a triangle is 10.7 centimeters, the slant height is 9.27 centimeters, and the height of the prism is 8.74 centimeters.

a. Find the area of the base of the tetrahedron.

b. Find the lateral area of the tetrahedron.

c. Find the total surface area of the tetrahedron.

d. Find the volume of the tetrahedron.

12. When Connie camps, she uses a tent that is in the form of a regular pyramid with a square base. The length of an edge of the base is 9 feet and the height of the tent at its center is 8 feet. Find the volume of the space enclosed by the tent.

13. Prove that the lateral edges of a regular pyramid with a base that is an equilateral triangle are congruent.

14. Let F be the vertex of a pyramid with square base $ABCD$. If $\overline{AF} \cong \overline{CF}$, prove that the pyramid is regular.

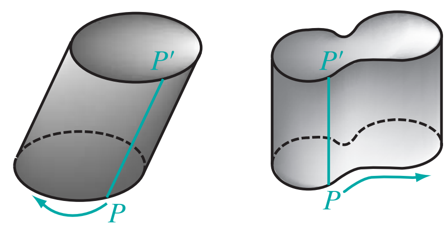
15. Prove that the altitudes of the lateral faces of a regular pyramid with a base that is an equilateral triangle are congruent.

16. Let p be the perimeter of the base of a regular pyramid and h_s be the slant height. Prove that the lateral area of a regular pyramid is equal to $\frac{1}{2}ph_s$.

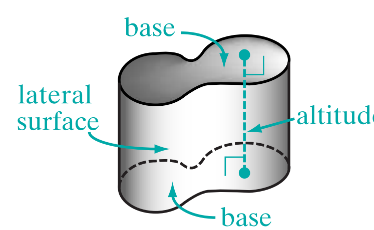
17. **a.** How does the lateral area of a regular pyramid change when both the slant height and the perimeter are doubled? tripled? Use the formula derived in exercise 16.

b. How does the volume of a regular pyramid with a triangle for a base change when both the sides of the base and the height of the pyramid are doubled? tripled?

11-7 CYLINDERS

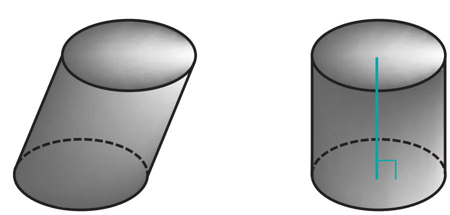


A prism has bases that are congruent polygons in parallel planes. What if the bases were congruent closed curves instead of polygons? Let $\overline{PP'}$ be a line segment joining corresponding points of two congruent curves. Imagine the surface generated as $\overline{PP'}$ moves along the curves, always joining corresponding points of the bases. The solid figure formed by the congruent parallel curves and the surface that joins them is called a **cylinder**.



The closed curves form the **bases** of the cylinder and the surface that joins the bases is the **lateral surface** of the cylinder. The **altitude** of a cylinder is a line segment perpendicular to the bases with endpoints on the bases. The **height** of a cylinder is the length of an altitude.

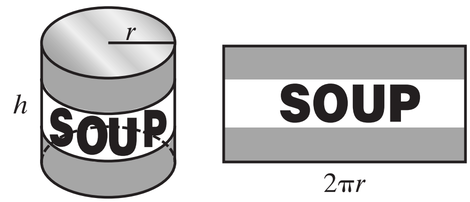
The most common cylinder is one that has bases that are congruent circles. This cylinder is a **circular cylinder**. If the line segment joining the centers of the circular bases is perpendicular to the bases, the cylinder is a **right circular cylinder**.



Circular cylinder

Right circular cylinder

Surface Area and Volume of a Circular Cylinder



The label on a cylindrical can of soup is a rectangle whose length is the circumference of the base of the can and whose width is the height of the can. This label is equal in area to the lateral surface of the cylindrical can. In Exercise 12 of Section 11-5, you proved that the lateral area, A, of a right prism is the product of the perimeter, p, of the prism and the height, h_p, of the prism ($A = ph_p$). The circumference of the base of a cylinder corresponds to the perimeter of the base of a prism. Therefore, we can say that the area of the lateral surface of a right circular cylinder is equal to the circumference of the circular base times the height of the cylinder.

If a right circular cylinder has bases that are circles of radius r and height h, then:

The lateral area of the cylinder = $2\pi rh$

The total surface area of the cylinder = $2\pi rh + 2\pi r^2$

The volume of the cylinder = $Bh = \pi r^2 h$

Note: The volume of *any* circular cylindar is $\pi r^2 h$.

EXAMPLE 1

Jenny wants to build a right circular cylinder out of cardboard with bases that have a radius of 6.0 centimeters and a height of 14 centimeters.

a. How many square centimeters of cardboard are needed for the cylinder *to the nearest square centimeter*?

b. What will be the volume of the cylinder *to the nearest cubic centimeter*?

Solution **a.**
$$\begin{aligned} A &= 2\pi rh + 2\pi r^2 \\ &= 2\pi(6.0)(14) + 2\pi(6.0)^2 \\ &= 168\pi + 72\pi \\ &= 240\pi \text{ cm}^2 \end{aligned}$$

Express this result as a rational approximation rounded to the nearest square centimeter.

$240\pi \approx 753.9822369 \approx 754 \text{ cm}^2$

b.
$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(6.0)^2(14) \\ &= 504\pi \text{ cm}^3 \end{aligned}$$

Express this result as a rational approximation rounded to the nearest cubic centimeter.

$504\pi \approx 1{,}583.362697 \approx 1{,}584 \text{ cm}^3$

Answers **a.** 754 cm^2 **b.** 1,584 cm^3

Exercises

Writing About Mathematics

1. Amy said that if the radius of a circular cylinder were doubled and the height decreased by one-half, the volume of the cylinder would remain unchanged. Do you agree with Amy? Explain why or why not.

2. Cindy said that if the radius of a right circular cylinder were doubled and the height decreased by one-half, the lateral area of the cylinder would remain unchanged. Do you agree with Cindy? Explain why or why not.

Developing Skills

In 3–6, the radius of a base, r, and the height, h, of a right circular cylinder are given. Find for each cylinder: **a.** the lateral area, **b.** the total surface area, **c.** the volume.

Express each measure as an exact value in terms of π.

3. $r = 34.0$ cm, $h = 60.0$ cm

4. $r = 4.0$ in., $h = 12$ in.

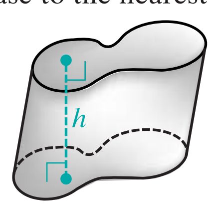
5. $r = 18.0$ in., $h = 2.00$ ft

6. $r = 1.00$ m, $h = 75.0$ cm

7. The volume of a right circular cylinder is 252 cubic centimeters and the radius of the base is 3.6 centimeters. What is the height of the cylinder to the nearest tenth?

8. The volume of a right circular cylinder is 586 cubic centimeters and the height of the cylinder is 4.6 centimeters. What is the radius of the base to the nearest tenth?

9. The areas of the bases of a cylinder are each 124 square inches and the volume of the cylinder is 1,116 cubic inches. What is the height, h, of the cylinder?



10. A circular cylinder has a base with a radius of 7.5 centimeters and a height of 12 centimeters. A rectangular prism has a square base and a height of 8.0 centimeters. If the cylinder and the prism have equal volumes, what is the length of the base of the prism to the nearest tenth?

Applying Skills

11. A can of beets has a top with a diameter of 2.9 inches and a height of 4.2 inches. What is the volume of the can to the nearest tenth?

12. A truck that delivers gasoline has a circular cylindrical storage space. The diameter of the bases of the cylinder is 11 feet, and the length (the height of the cylinder) is 17 feet. How many whole gallons of gasoline does the truck hold? (Use 1 cubic foot = 7.5 gallons.)

13. Karen makes pottery on a potter's wheel. Today she is making vases that are in the shape of a circular cylinder that is open at the top, that is, it has only one base. The base has a radius of 4.5 centimeters and is 0.75 centimeters thick. The lateral surface of the cylinder will be 0.4 centimeters thick. She uses 206 cubic centimeters of clay for each vase.

a. How much clay is used for the base of the vase to the nearest tenth?

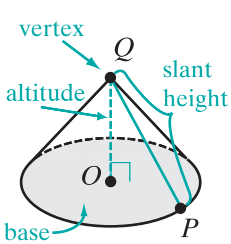
b. How much clay will be used for the lateral surface of the vase to the nearest tenth?

c. How tall will a vase be to the nearest tenth?

d. What will be the area of the lateral surface to the nearest tenth? (Use the value of the height of the vase found in part **c**.)

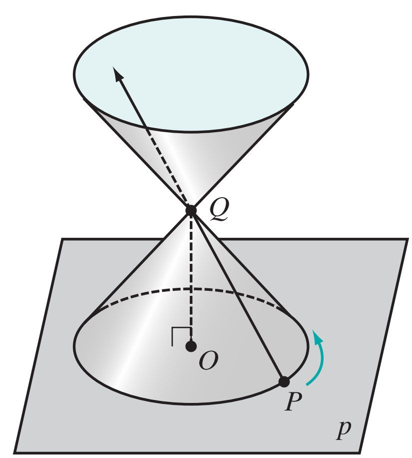
14. Mrs. Taggart sells basic cookie dough mix. She has been using circular cylindrical containers to package the mix but wants to change to rectangular prisms that will pack in cartons for shipping more efficiently. Her present packaging has a circular base with a diameter of 4.0 inches and a height of 5.8 inches. She wants the height of the new package to be 6.0 inches and the dimensions of the base to be in the ratio 2 : 5. Find, to the nearest tenth, the dimensions of the new package if the volume is to be the same as the volume of the cylindrical containers.

11-8 CONES



Think of $\overleftrightarrow{OQ}$ perpendicular to plane p at O. Think of a point P on plane p. Keeping point Q fixed, move P through a circle on p with center at O. The surface generated by $\overleftrightarrow{PQ}$ is a **right circular conical surface**. Note that a conical surface extends infinitely.

In our discussion, we will consider the part of the conical surface generated by $\overline{PQ}$ from plane p to Q, called a **right circular cone**. The point Q is the **vertex** of the cone. The circle in plane p with radius OP is the **base** of the cone, $\overline{OQ}$ is the **altitude** of the cone, OQ is the **height** of the cone, and PQ is the **slant height** of the cone.



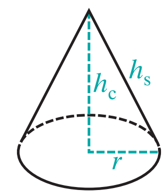
We can make a model of a right circular cone. Draw a large circle on a piece of paper and draw two radii. Cut out the circle and remove the part of the circle between the two radii. Join the two cut edges of the remaining part of the circle with tape.

Surface Area and Volume of a Cone

For a pyramid, we proved that the lateral area is equal to one-half the product of the perimeter of the base and the slant height. A similar relationship is true for a cone. The lateral area of a cone is equal to one-half the product of the circumference of the base and the slant height. Let L be the lateral area of the cone, C be the circumference of the base, S be the total surface area of the cone, h_s be the slant height, and r be the radius of the base. Then:

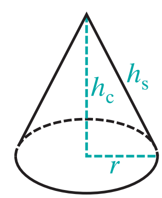
$$L = \frac{1}{2}Ch_s = \frac{1}{2}(2\pi r)h_s = \pi rh_s$$

$$S = L + \pi r^2 = \pi rh_s + \pi r^2$$



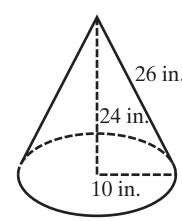
We can also use the relationship between the volume of a prism and the volume of a pyramid to write a formula for the volume of a cone. The volume of a cone is equal to one-third the product of the area of the base and the height of the cone. Let V be the volume of the cone, B be the area of the base with radius r, and h_c be the height of the cone. Then:

$$V = \frac{1}{3}Bh_c = \frac{1}{3}\pi r^2h_c$$



EXAMPLE 1

A right circular cone has a base with a radius of 10 inches, a height of 24 inches, and a slant height of 26 inches. Find the exact values of:



a. the lateral area **b.** the area of the base
c. the total surface area of the cone

Solution The radius of the base, r, is 10 inches. Therefore, the diameter of the base is 20 inches, and the circumference of the base, C, is 20π, the slant height, h_s, is 26 inches, and the height of the cone, h_c, is 24 inches.

a. Lateral area $= \frac{1}{2}Ch_s$

$= \frac{1}{2}(20\pi)(26)$

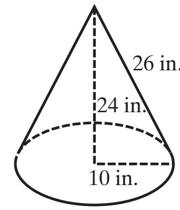
$= 260\pi$ in.^{2} *Answer*

b. Area of the base $= \pi(10)^2$

$= 100\pi \text{ cm}^2$ *Answer*

c. Total surface area $= 260\pi + 100\pi$

$= 360\pi \text{ in.}^2$ *Answer*



EXAMPLE 2

A cone and a cylinder have equal volumes and equal heights. If the radius of the base of the cone is 3 centimeters, what is the radius of the base of the cylinder?

Solution Let r be the radius of the base of the cylinder and h be the height of both the cylinder and the cone.

$$\text{Volume of the cylinder} = \pi r^2 h$$

$$\text{Volume of the cone} = \tfrac{1}{3}\pi(3)^2 h$$

$$\text{Volume of the cylinder} = \text{Volume of the cone}$$

$$\pi r^2 h = \tfrac{1}{3}\pi(3)^2 h$$

$$r^2 = 3$$

$$r = \sqrt{3}$$ *Answer*

Exercises

Writing About Mathematics

1. Elaine said that if a pyramid and a cone have equal heights and bases that have equal areas, then they have equal lateral areas. Do you agree with Elaine? Justify your answer.

2. Josephus said that if two cones have equal heights and the radius of one cone is equal to the diameter of the other, then the volume of the larger cone is twice the volume of the smaller. Do you agree with Josephus? Explain why or why not.

Developing Skills

In 3–6, the radius of the base, r, the slant height of the cone, h_s, and the height of the cone, h_c, are given. Find: **a.** the lateral area of the cone, **b.** the total surface area of the cone, **c.** the volume of the cone.

Express each measure as an exact value in terms of π and rounded to the nearest tenth.

3. $r = 3.00$ cm, $h_s = 5.00$ cm, $h_c = 4.00$ cm

4. $r = 5.00$ cm, $h_s = 13.0$ cm, $h_c = 12.0$ cm

5. $r = 24$ cm, $h_s = 25$ cm, $h_c = 7.0$ cm

6. $r = 8.00$ cm, $h_s = 10.0$ cm, $h_c = 6.00$ cm

7. The volume of a cone is 127 cubic inches and the height of the cone is 6.0 inches. What is the radius of the base to the nearest tenth?

8. The volume of a cone is 56 cubic centimeters and the area of the base is 48 square centimeters. What is the height of the cone to the nearest tenth?

9. The area of the base of a cone is equal to the area of the base of a cylinder, and their volumes are equal. If the height of the cylinder is 2 feet, what is the height of the cone?

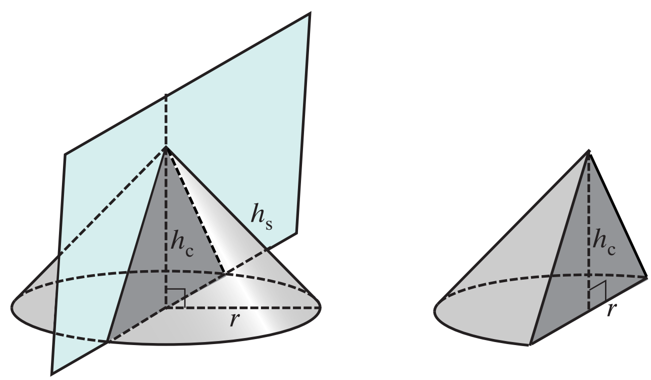
Applying Skills

10. The highway department has a supply of road salt for use in the coming winter. The salt forms a cone that has a height of 10 feet and a circular base with a diameter of 12 feet. How many cubic feet of salt does the department have stored? Round to the nearest foot.

11. The spire of the city hall is in the shape of a cone that has a circular base that is 20 feet in diameter. The slant height of the cone is 40 feet. How many whole gallons of paint will be needed to paint the spire if a gallon of paint will cover 350 square feet?

12. A cone with a height of 10 inches and a base with a radius of 6 inches is cut into two parts by a plane parallel to the base. The upper part is a cone with a height of 4 inches and a base with a radius of 2.4 inches. Find the volume of the lower part, the **frustum** of the cone, in terms of π.

13. When a cone is cut by a plane perpendicular to the base through the center of the base, the cut surface is a triangle whose base is the diameter of the base of the cone and whose altitude is the altitude of the cone. If the radius of the base is equal to the height of the cone, prove that the cut surface is an isosceles right triangle.



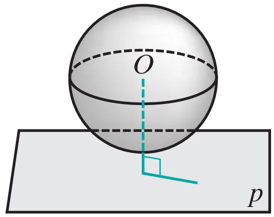
11-9 SPHERES

In a plane, the set of all points at a given distance from a fixed point is a circle. In space, this set of points is a *sphere*.

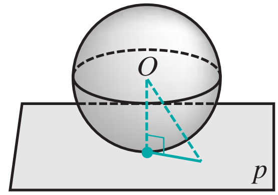
DEFINITION

A **sphere** is the set of all points equidistant from a fixed point called the **center**.

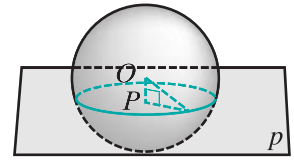
The **radius** of a sphere is the length of the line segment from the center of the sphere to any point on the sphere.



No points in common



1 point in common

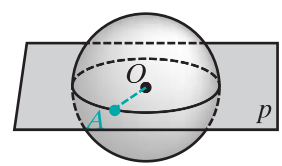


Infinitely many points in common

If the distance of a plane from the center of a sphere is greater than the radius of the sphere, the plane will have no points in common with the sphere. If the distance of a plane from the center of a sphere is equal to the radius of the sphere, the plane will have one point in common with the sphere. If the distance of a plane from the center of a sphere is less than the radius of the sphere, the plane will have infinitely many points in common with the sphere. We can prove that these points form a circle. Recall the definition of a circle.

► **A *circle* is the set of all points in a plane equidistant from a fixed point in the plane called the *center*.**

Theorem 11.14a The intersection of a sphere and a plane through the center of the sphere is a circle whose radius is equal to the radius of the sphere.



Given A sphere with center at O and radius r. Plane p through O intersects the sphere.

Prove The intersection is a circle with radius r.

Proof Let A be any point on the intersection. Since A is on the sphere, $OA = r$. Therefore, every point on the intersection is at the same distance from O and the intersection is a circle with radius r. ▫

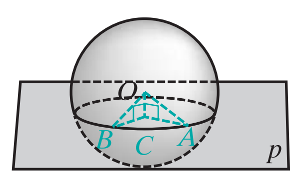
DEFINITION

A **great circle of a sphere** is the intersection of a sphere and a plane through the center of the sphere.

Theorem 11.14b If the intersection of a sphere and a plane does not contain the center of the sphere, then the intersection is a circle.

Given A sphere with center at O plane p intersecting the sphere at A and B.

Prove The intersection is a circle.



Proof

Statements	Reasons
1. Draw a line through O, perpendicular to plane p at C.	**1.** Through a given point there is one line perpendicular to a given plane.
2. $\angle OCA$ and $\angle OCB$ are right angles.	**2.** A line perpendicular to a plane is perpendicular to every line in the plane through the intersection of the line and the plane.
3. $\overline{OA} \cong \overline{OB}$	**3.** A sphere is the set of points in space equidistant from a fixed point.
4. $\overline{OC} \cong \overline{OC}$	**4.** Reflexive property.
5. $\triangle OAC \cong \triangle OBC$	**5.** HL.
6. $\overline{CA} \cong \overline{CB}$	**6.** Corresponding sides of congruent triangles are congruent.
7. The intersection is a circle.	**7.** A circle is the set of all points in a plane equidistant from a fixed point.

We can write Theorems 11.14a and 11.14b as a single theorem.

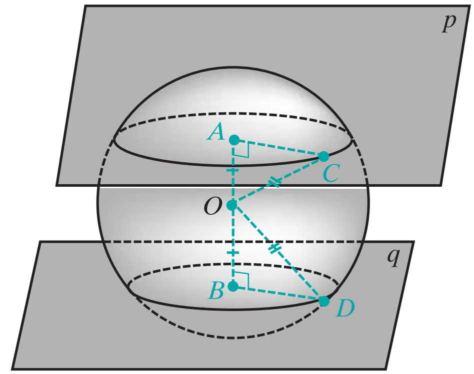
Theorem 11.14 The intersection of a plane and a sphere is a circle.

In the proof of Theorem 11.14b, we drew right triangle OAC with OA the radius of the sphere and AC the radius of the circle at which the plane and the sphere intersect. Since $\angle OCA$ is the right angle, it is the largest angle of $\triangle OAC$ and $OA > AC$. Therefore, a great circle, whose radius is equal to the radius of the sphere, is larger than any other circle that can be drawn on the sphere. We have just proved the following corollary:

Corollary 11.14a A great circle is the largest circle that can be drawn on a sphere.

Let p and q be any two planes that intersect the sphere with center at O. In the proof of Theorem 11.14b, the radius of the circle is the length of a leg of a right triangle whose hypotenuse is the radius of the sphere and whose other leg is the distance from the center of the circle to the plane. This suggests that if two planes are equidistant from the center of a sphere, they intersect the sphere in congruent circles.

Theorem 11.15 If two planes are equidistant from the center of a sphere and intersect the sphere, then the intersections are congruent circles.



Given A sphere with center at O intersected by planes p and q, $OA = OB$, $\overline{OA} \perp p$ and $\overline{OB} \perp q$.

Prove The intersections are congruent circles.

Proof Let C be any point on the intersection with p and D be any point on the intersection with q. Then $OA = OB$ and $OC = OD$ (they are both radii of the sphere). Therefore, $\triangle OAC$ and $\triangle OBD$ are congruent right triangles by HL. Since the corresponding sides of congruent triangles are congruent, the radii of the circles, AC and BD, are equal and the circles are congruent.

Surface Area and Volume of a Sphere

The formulas for the surface area and volume of a sphere are derived in advanced courses in mathematics. We can state and make use of these formulas.

The surface area of a sphere is equal to the area of four great circles. Let S be the surface area of a sphere of radius r. Then the surface area of the sphere is:

$$S = 4\pi r^2$$

The volume of a sphere is equal to four-thirds the product of π and the cube of the radius. Let V be the volume of a sphere of radius r. Then the volume of the sphere is:

$$V = \frac{4}{3}\pi r^3$$

EXAMPLE 1

Find the surface area and the volume of a sphere whose radius is 5.25 centimeters to the nearest centimeter.

Solution

$$\begin{aligned} S &= 4\pi r^2 \\ &= 4\pi(5.25)^2 \\ &= 110.25\pi \\ &\approx 346.3605901 \text{ cm}^2 \end{aligned}$$

When we round to the nearest centimeter to express the answer, $S = 346\text{ cm}^2$. *Answer*

$$\begin{aligned} V &= \tfrac{4}{3}\pi r^3 \\ &= \tfrac{4}{3}\pi(5.25)^3 \\ &= 192.9375\pi \\ &\approx 606.1310326\text{ cm}^3 \end{aligned}$$

When we round to the nearest centimeter to express the answer, $V = 606\text{ cm}^3$. *Answer*

Exercises

Writing About Mathematics

1. Meg said that if d is the diameter of a sphere, then the surface area of a sphere is equal to πd^2. Do you agree with Meg? Justify your answer.

2. Tim said that if the base a cone is congruent to a great circle of a sphere and the height of the cone is the radius of the sphere, then the volume of the cone is one-half the volume of the sphere. Do you agree with Tim? Justify your answer.

Developing Skills

In 3–6, find the surface area and the volume of each sphere whose radius, r, is given. Express each answer in terms of π and as a rational approximation to the nearest unit.

3. $r = 7.50$ in.

4. $r = 13.2$ cm

5. $r = 2.00$ ft

6. $r = 22.3$ cm

7. Find the radius of a sphere whose surface area is 100π square feet.

8. Find, to the nearest tenth, the radius of a sphere whose surface area is 84 square centimeters.

9. Find, to the nearest tenth, the radius of a sphere whose volume is 897 cubic inches.

10. Express, in terms of π, the volume of a sphere whose surface area is 196π square inches.

Applying Skills

11. A vase is in the shape of a sphere with a radius of 3 inches. How many whole cups of water will come closest to filling the vase? (1 cup = 14.4 cubic inches)

12. The radius of a ball is 5.0 inches. The ball is made of a soft foam that weighs 1 ounce per 40 cubic inches. How much does the ball weigh to the nearest tenth?

13. The diameter of the earth is about 7,960 miles. What is the surface area of the earth in terms of π?

14. The diameter of the moon is about 2,160 miles. What is the surface area of the moon in terms of π?

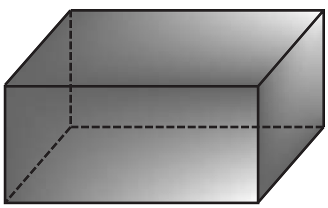
15. A cylinder has a base congruent to a great circle of a sphere and a height equal to the diameter of the sphere. If the radius of the sphere is 16 centimeters, compare the lateral area of the cylinder and the surface area of the sphere.

16. A cylinder has a base congruent to a great circle of a sphere and a height equal to the diameter of the sphere. If the diameter of the sphere is r, compare the lateral area of the cylinder and the surface area of the sphere.

Hands-On Activity

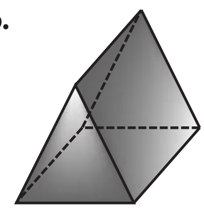
A **symmetry plane** is a plane that divides a solid into two congruent parts. For each solid shown below, determine the number of symmetry planes and describe their position relative to the solid.

a.



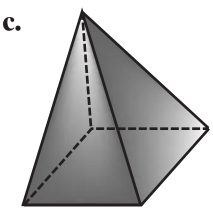
Rectangular prism

b.



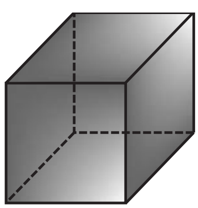
Prism with isosceles triangular base

c.



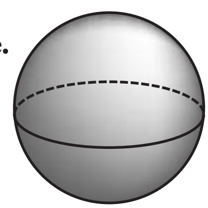
Regular pyramid with square base

d.



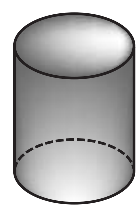
Cube

e.



Sphere

f.



Right circular cylinder

CHAPTER SUMMARY

Definitions to Know

- **Parallel lines in space** are lines in the same plane that have no points in common.
- **Skew lines** are lines in space that are neither parallel nor intersecting.
- A **dihedral angle** is the union of two half-planes with a common edge.
- The **measure of a dihedral angle** is the measure of the plane angle formed by two rays each in a different half-plane of the angle and each perpendicular to the common edge at the same point of the edge.

- **Perpendicular planes** are two planes that intersect to form a right dihedral angle.
- A **line is perpendicular to a plane** if and only if it is perpendicular to each line in the plane through the intersection of the line and the plane.
- A **plane is perpendicular to a line** if the line is perpendicular to the plane.
- **Parallel planes** are planes that have no points in common.
- A **line is parallel to a plane** if it has no points in common with the plane.
- The **distance between two planes** is the length of the line segment perpendicular to both planes with an endpoint on each plane.
- A **polyhedron** is a three-dimensional figure formed by the union of the surfaces enclosed by plane figures.
- The portions of the planes enclosed by a plane figure are called the **faces** of the polyhedron.
- The intersections of the faces are the **edges** of the polyhedron, and the intersections of the edges are the **vertices** of the polyhedron.
- A **prism** is a polyhedron in which two of the faces, called the **bases** of the prism, are congruent polygons in parallel planes.
- The sides of a prism that are not bases are called the **lateral sides**.
- The union of two lateral sides is a **lateral edge**.
- If the line segments joining the corresponding vertices of the bases of a prism are perpendicular to the planes of the bases, then the prism is a **right prism**.
- The **altitude** of a prism is a line segment perpendicular to each of the bases with an endpoint on each base.
- The **height** of a prism is the length of an altitude.
- A **parallelepiped** is a prism that has parallelograms as bases.
- A **rectangular parallelepiped** is a parallelepiped that has rectangular bases and lateral edges perpendicular to the bases.
- The **lateral area** of the prism is the sum of the areas of the lateral faces.
- The **total surface area** of a solid figure is the sum of the lateral area and the areas of the bases.
- A **pyramid** is a solid figure with a base that is a polygon and lateral faces that are triangles.
- A **regular pyramid** is a pyramid whose base is a regular polygon and whose altitude is perpendicular to the base at its center.
- The length of the altitude of a triangular lateral face of a regular pyramid is the **slant height** of the pyramid.
- A **cylinder** is a solid figure formed by congruent parallel curves and the surface that joins them.

- If the line segment joining the centers of circular bases of a cylinder is perpendicular to the bases, the cylinder is a **right circular cylinder**.
- Let $\overline{OP}$ be a line segment perpendicular to a plane at O and A be a point on a circle in the plane with center at O. A **right circular cone** is the solid figure that is the union of a circular base and the surface generated by line segment $\overline{AP}$ as P moves around the circle.
- A **sphere** is the set of all points equidistant from a fixed point called the **center**.
- The **radius** of a sphere is the length of the line segment from the center of the sphere to any point on the sphere.
- A **great circle of a sphere** is the intersection of a sphere and a plane through the center of the sphere.

Postulates

11.1 There is one and only one plane containing three non-collinear points.
11.2 A plane containing any two points contains all of the points on the line determined by those two points.
11.3 If two planes intersect, then they intersect in exactly one line.
11.4 At a given point on a line, there are infinitely many lines perpendicular to the given line.
11.5 The volume of a prism is equal to the area of the base times the height.

Theorems and Corollaries

11.1 There is exactly one plane containing a line and a point not on the line.
11.2 Two intersecting lines determine a plane.
11.3 If a line not in a plane intersects the plane, then it intersects in exactly one point.
11.4 If a line is perpendicular to each of two intersecting lines at their point of intersection, then the line is perpendicular to the plane determined by these lines.
11.5 Two planes are perpendicular if and only if one plane contains a line perpendicular to the other.
11.6 Through a given point on a plane, there is only one line perpendicular to the given plane.
11.7 Through a given point on a line, there can be only one plane perpendicular to the given line.
11.8 If a line is perpendicular to a plane, then any line perpendicular to the given line at its point of intersection with the given plane is in the plane.
11.9 If a line is perpendicular to a plane, then every plane containing the line is perpendicular to the given plane.
11.10 If a plane intersects two parallel planes, then the intersection is two parallel lines.
11.11 Two lines perpendicular to the same plane are parallel.
11.11a Two lines perpendicular to the same plane are coplanar.
11.12 Two planes are perpendicular to the same line if and only if the planes are parallel.
11.13 Parallel planes are everywhere equidistant.

11.14 The intersection of a plane and a sphere is a circle.
11.14a A great circle is the largest circle that can be drawn on a sphere.
11.15 If two planes are equidistant from the center of a sphere and intersect the sphere, then the intersections are congruent circles.

Formulas

L = lateral area
S = surface area
C = circumference of the base of a cone
V = volume
p = perimeter of a base
r = radius of a base of a cylinder or a cone; radius of a sphere
B = area of a base
h_s = height of a lateral surface
h_p = height of a prism or a pyramid
h_c = height of a cylinder or a cone

Right Prism	Regular Pyramid	Right Circular Cylinder
$L = ph_s$	$L = \frac{1}{2}ph_s$	$L = 2\pi rh_c$
$S = L + 2B$	$S = L + B$	$S = 2\pi rh_c + 2\pi r^2$
$V = Bh_p$	$V = \frac{1}{3}Bh_p$	$V = Bh_c = \pi r^2h_c$

Cone	Sphere
$L = \frac{1}{2}Ch_s = \pi rh_s$	$S = 4\pi r^2$
$V = \frac{1}{3}Bh_c = \frac{1}{3}\pi r^2h_c$	$V = \frac{4}{3}\pi r^3$

VOCABULARY

11-1 Cavalieri's Principle • Solid geometry • Parallel lines in space • Skew lines

11-2 Dihedral angle • Plane angle • Measure of a dihedral angle • Perpendicular planes • Line perpendicular to a plane • Plane perpendicular to a line

11-3 Parallel planes • Line parallel to a plane • Distance between two planes

11-4 Polyhedron • Faces of a polyhedron • Edges of a polyhedron • Vertices of a polyhedron • Prism • Bases of a prism • Lateral sides of a prism • Lateral edge of a prism • Altitude of a prism • Height of a prism • Right prism • Parallelepiped • Rectangular parallelepiped • Rectangular solid • Lateral area • Total surface area

11-5 Cubic centimeter

11-6 Pyramid • Vertex of a pyramid • Altitude of a pyramid • Height of a pyramid • Regular pyramid • Slant height of a pyramid

11-7 Cylinder • Base of a cylinder • Lateral surface of a cylinder • Altitude of a cylinder • Height of a cylinder • Right circular cylinder

11-8 Right circular conical surface • Right circular cone • Vertex of a cone • Base of a cone • Altitude of a cone • Height of a cone • Slant height of a cone • Frustum of a cone

11-9 Sphere • Center of a sphere • Radius of a sphere • Great circle of a sphere • Symmetry plane

REVIEW EXERCISES

In 1–16, answer each question and state the postulate, theorem, or definition that justifies your answer or draw a counterexample.

1. Lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect at E. A line, $\overleftrightarrow{EF}$, is perpendicular to $\overleftrightarrow{AB}$ and to $\overleftrightarrow{CD}$. Is $\overleftrightarrow{EF}$ perpendicular to the plane determined by the intersecting lines?

2. Plane p is perpendicular to $\overleftrightarrow{AB}$ at B. Plane q intersects $\overleftrightarrow{AB}$ at B. Can q be perpendicular to $\overleftrightarrow{AB}$?

3. A line, $\overleftrightarrow{RS}$, is perpendicular to plane p at R. If T is a second point not on p, can $\overleftrightarrow{RT}$ be perpendicular to plane p?

4. Lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{LM}$ are each perpendicular to plane p. Are $\overleftrightarrow{AB}$ and $\overleftrightarrow{LM}$ coplanar?

5. A line $\overleftrightarrow{AB}$ is in plane q and $\overleftrightarrow{AB}$ is perpendicular to plane p. Are planes p and q perpendicular?

6. Two planes, p and q, are perpendicular to each other. Does p contain a line perpendicular to q?

7. A line $\overleftrightarrow{AB}$ is perpendicular to plane p at B and $\overleftrightarrow{BC} \perp \overleftrightarrow{AB}$. Is $\overleftrightarrow{BC}$ in plane p?

8. A line $\overleftrightarrow{RS}$ is perpendicular to plane p and $\overleftrightarrow{RS}$ is in plane q. Is plane p perpendicular to plane q?

9. Plane r intersects plane p in $\overleftrightarrow{AB}$ and plane r intersects plane q in $\overleftrightarrow{CD}$. Can $\overleftrightarrow{AB}$ intersect $\overleftrightarrow{CD}$?

10. Planes p and q are each perpendicular to $\overleftrightarrow{AB}$. Are p and q parallel?

11. Two lateral edges of a prism are $\overline{AB}$ and $\overline{CD}$. Can $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect?

12. Two lateral edges of a prism are $\overline{AB}$ and $\overline{CD}$. Is $AB = CD$?

13. Two prisms have equal heights and bases with equal areas. Do the prisms have equal volumes?

14. A prism and a pyramid have equal heights and bases with equal areas. Do the prism and the pyramid have equal volumes?

15. Two planes intersect a sphere at equal distances from the center of the sphere. Are the circles at which the planes intersect the sphere congruent?

16. Plane *p* intersects a sphere 2 centimeters from the center of the sphere and plane *q* contains the center of the sphere and intersects the sphere. Are the circles at which the planes intersect the sphere congruent circles?

In 17–22, find the lateral area, total surface area, and volume of each solid figure to the nearest unit.

17. The length of each side of the square base of a rectangular prism is 8 centimeters and the height is 12 centimeters.

18. The height of a prism with bases that are right triangles is 5 inches. The lengths of the sides of the bases are 9, 12, and 15 inches.

19. The base of a rectangular solid measures 9 feet by 7 feet and the height of the solid is 4 feet.

20. A pyramid has a square base with an edge that measures 6 inches. The slant height of a lateral side is 5 inches and the height of the pyramid is 4 inches.

21. The diameter of the base of a cone is 10 feet, its height is 12 feet, and its slant height is 13 feet.

22. The radius of the base of a right circular cylinder is 7 centimeters and the height of the cylinder is 9 centimeters.

23. A cone and a pyramid have equal volumes and equal heights. Each side of the square base of the pyramid measures 5 meters. What is the radius of the base of the cone? Round to the nearest tenth.

24. Two prisms with square bases have equal volumes. The height of one prism is twice the height of the other. If the measure of a side of the base of the prism with the shorter height is 14 centimeters, find the measure of a side of the base of the other prism in simplest radical form.

25. Ice cream is sold by street vendors in containers that are right circular cylinders. The base of the cylinder has a diameter of 5 inches and the cylinder has a height of 6 inches.

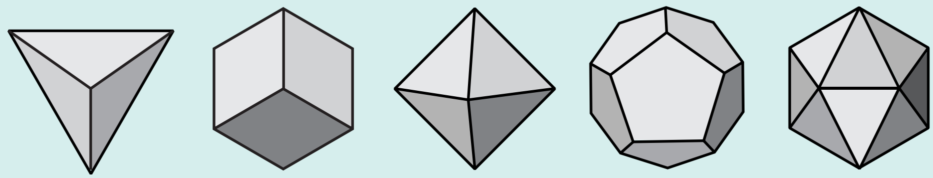
a. Find, to the nearest tenth, the amount of ice cream that a container can hold.

b. If a scoop of ice cream is a sphere with a diameter of 2.4 inches, find, to the nearest tenth, the amount of ice cream in a single scoop.

c. If the ice cream is packed down into the container, how many whole scoops of ice cream will come closest to filling the container?

Exploration

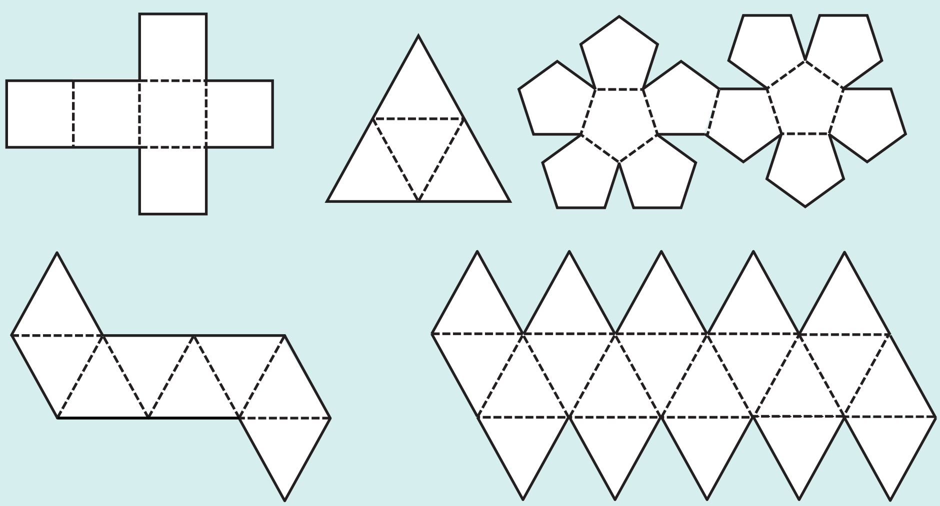
A **regular polyhedron** is a solid, all of whose faces are congruent regular polygons with the sides of the same number of polygons meeting at each vertex. There are five regular polyhedra: a tetrahedron, a cube, an octahedron, a dodecahedron, and an icosahedron. These regular polyhedra are called the **Platonic solids**.



Tetrahedron Cube Octahedron Dodecahedron Icosahedron

a. Make a paper model of the Platonic solids as follows:
(1) Draw the diagrams below on paper.
(2) Cut out the diagrams along the solid lines and fold along the dotted lines.
(3) Tape the folded sides together.

Note: You may wish to enlarge to the diagrams for easier folding.



b. Using the solids you constructed in part **a**, fill in the table below:

	Number of vertices	Number of faces	Number of edges
Tetrahedron			
Cube			
Octahedron			
Dodecahedron			
Icosahedron			

Do you observe a relationship among the number of vertices, faces, and edges for each Platonic solid? If so, state this relationship.

c. Research the five Platonic solids and investigate why there are only five regular polyhedra. Share your findings with your classmates.

CUMULATIVE REVIEW Chapters 1–11

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. If the measures of the angles of a triangle are represented by x, $2x - 20$, and $2x$, what is the measure of the smallest angle?

(1) 40 (2) 60 (3) 80 (4) 90

2. In $\triangle ABC$, $m\angle A = 40$ and the measure of an exterior angle at B is 130. The triangle is

(1) scalene and acute
(2) scalene and right
(3) isosceles and right
(4) isosceles and acute

3. The coordinates of the midpoint of a segment with endpoints at (2, –3) and (−6, 1) are

(1) (4, 2) (2) (4, −2) (3) (−4, −2) (4) (−2, −1)

4. The lengths of the diagonals of a rhombus are 8 centimeters and 12 centimeters. The area of the rhombus is

(1) 24 cm^2 (2) 32 cm^2 (3) 48 cm^2 (4) 96 cm^2

5. Two parallel lines are cut by a transversal. The measure of one interior angle is $x + 7$ and the measure of another interior angle on the same side of the transversal is $3x - 3$. What is the value of x?

(1) 5 (2) 12 (3) 44 (4) 51

6. If "Today is Monday" is true and "It is May 5" is false, which of the following is true?
(1) Today is Monday and it is May 5.
(2) If today is Monday, then it is May 5.
(3) Today is Monday only if it is May 5.
(4) Today is Monday or it is May 5.

7. Which of the following do not always lie in the same plane?
(1) three points
(2) two parallel lines
(3) two intersecting lines
(4) three parallel lines

8. What is the slope of a line perpendicular to the line whose equation is $x - 2y = 3$?
(1) $\frac{1}{2}$ (2) 2 (3) $-\frac{1}{2}$ (4) -2

9. A quadrilateral has diagonals that are not congruent and are perpendicular bisectors of each other. The quadrilateral is a
(1) square (2) rectangle (3) trapezoid (4) rhombus

10. The base of a right prism is a square whose area is 36 square centimeters. The height of the prism is 5 centimeters. The lateral area of a prism is
(1) 30 cm^2 (2) 60 cm^2 (3) 120 cm^2 (4) 180 cm^2

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. A leg, $\overline{AB}$, of isosceles $\triangle ABC$ is congruent to a leg, $\overline{DE}$, of isosceles $\triangle DEF$. The vertex angle, $\angle B$, of isosceles $\triangle ABC$ is congruent to the vertex angle, $\angle E$, of isosceles $\triangle DEF$. Prove that $\triangle ABC \cong \triangle DEF$.

12. In triangle ABC, altitude $\overline{CD}$ bisects $\angle C$. Prove that the triangle is isosceles.

Part III

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. Quadrilateral $BCDE$ is a parallelogram and B is the midpoint of $\overline{ABC}$. Prove that $ABDE$ is a parallelogram.

14. Line segment $\overline{ABC}$ is perpendicular to plane p at B, the midpoint of $\overline{ABC}$. Prove that any point on p is equidistant from A and C.

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. Write the equation of the median from B to $\overline{AC}$ of $\triangle ABC$ if the coordinates of the vertices are $A(-3, -2)$, $B(-1, 4)$, and $C(5, -2)$.

16. The coordinates of the endpoints of $\overline{AB}$ are $A(1, 3)$ and $B(5, -1)$. Find the coordinates of the endpoints of $\overline{A'B'}$, the image of $\overline{AB}$ under the composition $r_{x\text{-axis}} \circ R_{90°}$.