CHAPTER

8

TRIGONOMETRY OF THE RIGHT TRIANGLE

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The accurate measurement of land has been a critical challenge throughout the history of civilization. Today's land measurement problems are not unlike those George Washington might have solved by using measurements made with a transit, but the modern surveyor has available a total workstation including EDM (electronic distance measuring) and a theodolite for angle measurement. Although modern instruments can perform many measurements and calculations, the surveyor needs to understand the principles of indirect measurement and trigonometry to correctly interpret and apply these results.

In this chapter, we will begin the study of a branch of mathematics called **trigonometry**. The word *trigonometry* is Greek in origin and means "measurement of triangles." Although the trigonometric functions have applications beyond the study of triangles, in this chapter we will limit the applications to the study of right triangles.

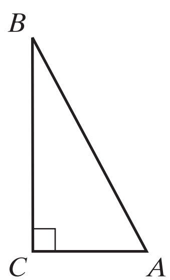
8-1 THE PYTHAGOREAN THEOREM

The solutions of many problems require the measurement of line segments and angles. When we use a ruler or tape measure to determine the length of a segment, or a protractor to find the measure of an angle, we are taking a **direct measurement** of the segment or the angle. In many situations, however, it is inconvenient or impossible to make a measurement directly. For example, it is difficult to make the direct measurements needed to answer the following questions:

What is the height of a 100-year-old oak tree?
What is the width of a river?
What is the distance to the sun?

We can answer these questions by using methods that involve **indirect measurement**. Starting with some known lengths of segments or angle measures, we apply a formula or a mathematical relationship to indirectly find the measurement in question.

Engineers, surveyors, physicists, and astronomers frequently use these trigonometric methods in their work.

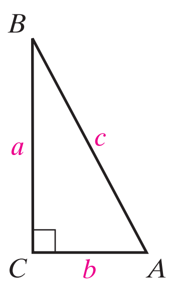


The figure at the left represents a *right triangle*. Recall that such a triangle contains one and only one right angle. In right triangle ABC, side $\overline{AB}$, which is opposite the right angle, is called the *hypotenuse*.

The hypotenuse is the longest side of the triangle. The other two sides of the triangle, $\overline{BC}$ and $\overline{AC}$, form the right angle. They are called the *legs* of the right triangle.

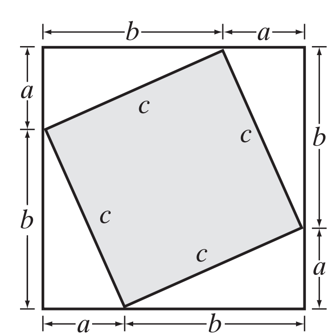
More than 2,000 years ago, the Greek mathematician Pythagoras demonstrated the following property of the right triangle, which is called the **Pythagorean Theorem**:

> **The Pythagorean Theorem** In a right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.



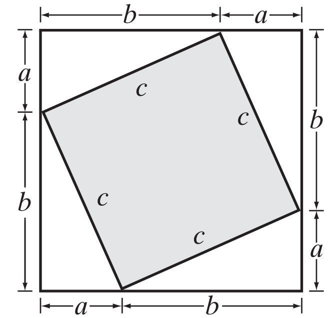
If we represent the length of the hypotenuse of right triangle ABC by c and the lengths of the other two sides by a and b, the Theorem of Pythagoras may be written as the following formula:

$$c^2 = a^2 + b^2$$



To show that this relationship is true for any right triangle ABC with the length of the hypotenuse represented by c and the lengths of the legs represented by a and b, consider a square with sides $(a + b)$.

The area of the square is $(a + b)^2$. However, since it is divided into four triangles and one smaller square, its area can also be expressed as:



$$\begin{aligned}\text{Area of the square} &= \text{area of the four triangles} \\ &\quad + \text{area of the smaller square} \\ &= 4\left(\tfrac{1}{2}\right)ab + c^2\end{aligned}$$

Although the area is written in two different ways, both expressions are equal.

Thus,

$$(a + b)^2 = 4\left(\tfrac{1}{2}\right)ab + c^2.$$

If we simplify, we obtain the relationship of the Pythagorean Theorem:

$$(a + b)^2 = 4\left(\tfrac{1}{2}\right)ab + c^2$$

$$a^2 + 2ab + b^2 = 2ab + c^2$$ Expand the binomial term $(a + b)^2$.

$$a^2 + 2ab - 2ab + b^2 = 2ab - 2ab + c^2$$ Subtract $2ab$ from both sides of the equality.

$$a^2 + b^2 = c^2$$

Statements of the Pythagorean Theorem

Two statements can be made for any right triangle where c represents the length of the hypotenuse (the longest side) and a and b represent the lengths of the other two sides.

1. If a triangle is a right triangle, then the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides. If a triangle is a right triangle, then $c^2 = a^2 + b^2$.
2. If the square of the length of the longest side of a triangle is equal to the sum of the squares of the lengths of the other two sides, the triangle is a right triangle. If $c^2 = a^2 + b^2$ in a triangle, then the triangle is a right triangle.

If we know the lengths of any two sides of a right triangle, we can find the length of the third side. For example, if the measures of the legs of a right triangle are 7 and 9, we can write:

$$\begin{aligned}c^2 &= a^2 + b^2 \\ c^2 &= 7^2 + 9^2 \\ c^2 &= 49 + 81 \\ c^2 &= 130\end{aligned}$$

To solve this equation for c, we must do the opposite of squaring, that is, we must find the square root of 130. There are two square roots of 130, $+\sqrt{130}$ and $-\sqrt{130}$ which we write as $\pm\sqrt{130}$.

There are two things that we must consider here when finding the value of c.

1. Since c represents the length of a line segment, only the positive number is an acceptable value. Therefore, $c = +\sqrt{130}$.
2. There is no rational number that has a square of 130. The value of c is an irrational number. However, we usually use a calculator to find a rational approximation for the irrational number.

ENTER: **2nd** **√** 130 **ENTER**

DISPLAY:

```
√(130
          11.40175425
```

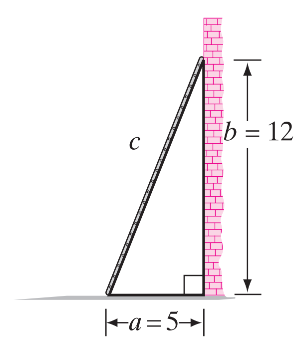
Therefore, to the nearest tenth, the length of the hypotenuse is 11.4. Note that the calculator gives only the positive rational approximation of the square root of 130.

EXAMPLE 1

A ladder is placed 5 feet from the foot of a wall. The top of the ladder reaches a point 12 feet above the ground. Find the length of the ladder.

Solution The ladder, the wall, and the ground form a right triangle. The length of the ladder is c, the length of the hypotenuse of the right triangle. The distance from the foot of the ladder to the wall is $a = 5$, and the distance from the ground to the top of the ladder is $b = 12$. Use the Theorem of Pythagoras.

$$c^2 = a^2 + b^2$$
$$c^2 = 5^2 + 12^2$$
$$c^2 = 25 + 144$$
$$c^2 = 169$$
$$c = \pm\sqrt{169} = \pm 13$$



Reject the negative value. Note that in this case the exact value of c is a rational number because 169 is a perfect square.

Answer The length of the ladder is 13 feet.

EXAMPLE 2

The hypotenuse of a right triangle is 36.0 centimeters long and one leg is 28.5 centimeters long.

a. Find the length of the other leg to the nearest tenth of a centimeter.

b. Find the area of the triangle using the correct number of significant digits.

Solution **a.** The length of the hypotenuse is $c = 36.0$ and the length of one leg is $a = 28.5$. The length of the other leg is b. Substitute the known values in the Pythagorean Theorem.

$$\begin{aligned} c^2 &= a^2 + b^2 \\ 36.0^2 &= 28.5^2 + b^2 \\ 1{,}296 &= 812.25 + b^2 \\ 483.75 &= b^2 \\ \pm\sqrt{483.75} &= b \end{aligned}$$

Reject the negative value. Use a calculator to find a rational approximation of the value of b. A calculator displays 21.99431745. Round the answer to the nearest tenth.

Answer The length of the other leg is 22.0 centimeters.

b. Area of $\triangle ABC = \frac{1}{2}bh = \frac{1}{2}(28.5)(22.0) = 313.5$

Since the lengths are given to three significant digits, we will round the area to three significant digits.

Answer The area of $\triangle ABC$ is 314 square centimeters.

EXAMPLE 3

Is a triangle whose sides measure 8 centimeters, 7 centimeters, and 4 centimeters a right triangle?

Solution If the triangle is a right triangle, the longest side, whose measure is 8, must be the hypotenuse. Then:

$$\begin{aligned} c^2 &= a^2 + b^2 \\ 8^2 &\stackrel{?}{=} 7^2 + 4^2 \\ 64 &\stackrel{?}{=} 49 + 16 \\ 64 &\neq 65 \ ✘ \end{aligned}$$

Answer The triangle is not a right triangle.

EXERCISES

Writing About Mathematics

1. A **Pythagorean triple** is a set of three positive integers that make the equation $c^2 = a^2 + b^2$ true. Luz said that 3, 4, and 5 is a Pythagorean triple, and, for any positive integer k, $3k$, $4k$, and $5k$ is also a Pythagorean triple. Do you agree with Luz? Explain why or why not.

2. Regina said that if n is a positive integer, $2n + 1$, $2n^2 + 2n$, and $2n^2 + 2n + 1$ is a Pythagorean triple. Do you agree with Regina? Explain why or why not.

Developing Skills

In 3–11, c represents the length of the hypotenuse of a right triangle and a and b represent the lengths of the legs. For each right triangle, find the length of the side whose measure is not given.

3. $a = 3, b = 4$ **4.** $a = 8, b = 15$ **5.** $c = 10, a = 6$

6. $c = 13, a = 12$ **7.** $c = 17, b = 15$ **8.** $c = 25, b = 20$

9. $a = \sqrt{2}, b = \sqrt{2}$ **10.** $a = 1, b = \sqrt{3}$ **11.** $a = \sqrt{8}, c = 3$

In 12–17, c represents the length of the hypotenuse of a right triangle and a and b represent the lengths of the legs. For each right triangle:

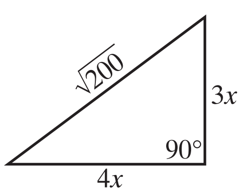
a. Express the length of the third side in radical form.

b. Express the length of the third side to the *nearest hundredth*.

12. $a = 2, b = 3$ **13.** $a = 3, b = 3$ **14.** $a = 4, c = 8$

15. $a = 7, b = 2$ **16.** $b = \sqrt{3}, c = \sqrt{14}$ **17.** $a = \sqrt{7}, c = 6$

In 18–21, find x in each case and express irrational results in radical form.

18.


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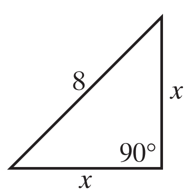

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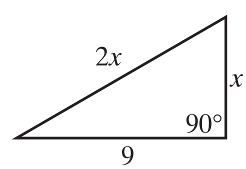

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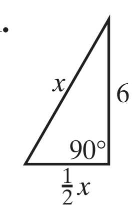


In 22–27, find, in each case, the length of the diagonal of a rectangle whose sides have the given measurements.

22. 7 inches by 24 inches **23.** 9 centimeters by 40 centimeters

24. 28 feet by 45 feet **25.** 17 meters by 144 meters

26. 15 yards by 20 yards **27.** 18 millimeters by 24 millimeters

28. The diagonal of a rectangle measures 65 centimeters. The length of the rectangle is 33 centimeters. Consider the measurements to be exact.

a. Find the width of the rectangle.

b. Find the area of the rectangle.

29. Approximate, to the nearest inch, the length of a rectangle whose diagonal measures 25.0 inches and whose width is 18.0 inches.

30. The altitude to the base of a triangle measures 17.6 meters. The altitude divides the base into two parts that are 12.3 meters and 15.6 meters long. What is the perimeter of the triangle to the nearest tenth of a meter?

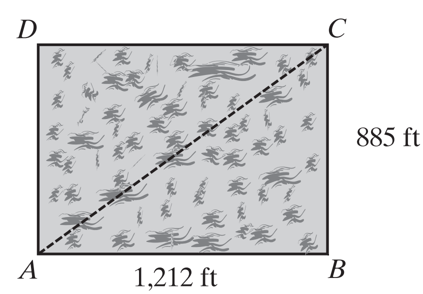
Applying Skills

31. A ladder 39 feet long leans against a building and reaches the ledge of a window. If the foot of the ladder is 15 feet from the foot of the building, how high is the window ledge above the ground to the nearest foot?

32. Mr. Rizzo placed a ladder so that it reached a window 15.0 feet above the ground when the foot of the ladder was 5.0 feet from the wall. Find the length of the ladder to the nearest tenth of a foot.

33. Mrs. Culkowski traveled 24.0 kilometers north and then 10.0 kilometers east. How far was she from her starting point?

34. One day, Ronnie left his home at A and reached his school at C by walking along $\overline{AB}$ and $\overline{BC}$, the sides of a rectangular open field that was muddy. The dimensions of the field are 1,212 feet by 885 feet. When he was ready to return home, the field was dry and Ronnie decided to take a shortcut by walking diagonally across the field, along $\overline{AC}$. To the nearest whole foot, how much shorter was the trip home than the trip to school?



35. Corry and Torry have a two-way communication device that has a range of one-half mile (2,640 feet). Torry lives 3 blocks west and 2 blocks north of Corry. If the length of each block is 600 feet, can Corry and Torry communicate using this device when each is home? Explain your answer.

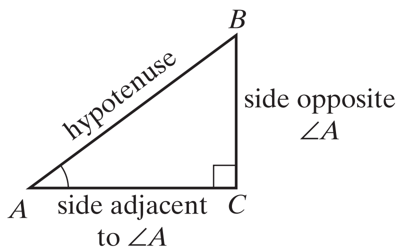
36. A baseball diamond has the shape of a square with the bases at the vertices of the square. If the distance from home plate to first base is 90.0 feet, approximate, to the nearest tenth of a foot, the distance from home plate to second base.

8-2 THE TANGENT RATIO

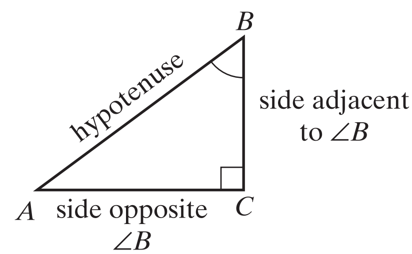
Naming Sides

In a right triangle, the hypotenuse, which is the longest side, is opposite the right angle. The other two sides in a right triangle are called the legs. However, in trigonometry of the right triangle, we call these legs the **opposite side** and the **adjacent side** to describe their relationship to one of the acute angles in the triangle.

Notice that $\triangle ABC$ is the same right triangle in both figures below, but the position names we apply to the legs change with respect to the angles.



In $\triangle ABC$:
$\overline{BC}$ is *opposite* $\angle A$;
$\overline{AC}$ is *adjacent* to $\angle A$.



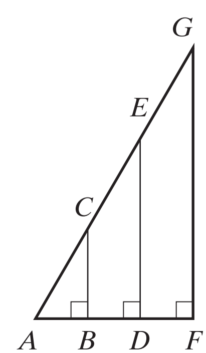
In $\triangle ABC$:
$\overline{AC}$ is *opposite* $\angle B$;
$\overline{BC}$ is *adjacent* to $\angle B$.

Similar Triangles

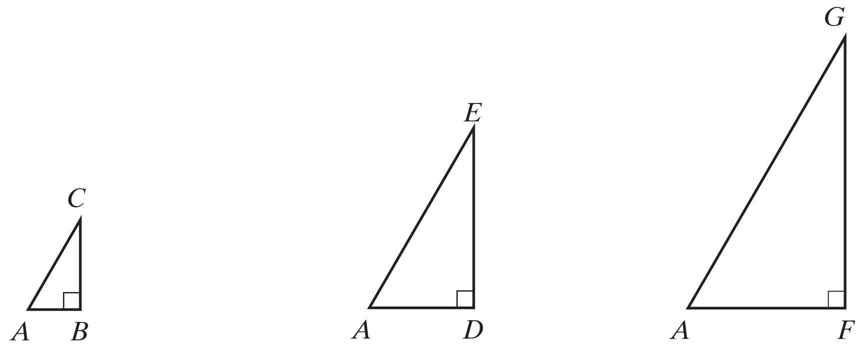
Three right triangles are drawn to coincide at vertex A. Since each triangle contains a right angle as well as $\angle A$, we know that the third angles of each triangle are congruent. When three angles of one triangle are congruent to the three angles of another, the triangles are **similar**.

The corresponding sides of similar triangles are in proportion. Therefore:

$$\frac{CB}{BA} = \frac{ED}{DA} = \frac{GF}{FA}$$



The similar triangles shown in the previous page, $\triangle ABC$, $\triangle ADE$, and $\triangle AFG$, are separated and shown below.



In $\triangle ABC$:
$\overline{CB}$ is opposite $\angle A$;
$\overline{BA}$ is adjacent to $\angle A$.

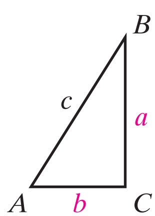
In $\triangle ADE$:
$\overline{ED}$ is opposite $\angle A$;
$\overline{DA}$ is adjacent to $\angle A$.

In $\triangle AFG$:
$\overline{GF}$ is opposite $\angle A$;
$\overline{FA}$ is adjacent to $\angle A$.

Therefore, $\frac{CB}{BA} = \frac{ED}{DA} = \frac{GF}{FA} = \frac{\text{length of side } \textit{opposite} \angle A}{\text{length of side } \textit{adjacent} \text{ to } \angle A}$ = a constant for $\triangle ABC$, $\triangle ADE$, $\triangle AFG$ and for any right triangle similar to these triangles, that is, for any right triangle with an acute angle congruent to $\angle A$. This ratio is called the **tangent of the angle**.

DEFINITION

The **tangent of an acute angle of a right triangle** is the ratio of the length of the side opposite the acute angle to the length of the side adjacent to the acute angle.



For right triangle ABC, with $m\angle C = 90$, the definition of the *tangent of* $\angle A$ is as follows:

$$\textbf{tangent } A = \frac{\text{length of side } \textit{opposite} \angle A}{\text{length of side } \textit{adjacent} \text{ to } \angle A} = \frac{BC}{AC} = \frac{a}{b}$$

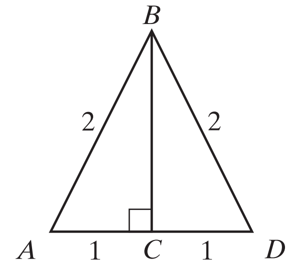
By using "tan A" as an abbreviation for tangent A, "opp" as an abbreviation for the length of the leg opposite $\angle A$, and "adj" as an abbreviation for the length of the leg adjacent to $\angle A$, we can shorten the way we write the relationship given above as follows:

$$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AC} = \frac{a}{b}$$

Finding Tangent Ratios on a Calculator

The length of each side of equilateral triangle ABD is 2. The altitude $\overline{BC}$ from B to $\overline{AD}$ forms two congruent right triangles with $AC = 1$. We can use the hypotenuse rule to find BC.

$$(AC)^2 + (BC)^2 = (AB)^2$$
$$1^2 + (BC)^2 = 2^2$$
$$1 + (BC)^2 = 4$$
$$(BC)^2 = 3$$
$$BC = \sqrt{3}$$



The measure of each angle of an equilateral triangle is 60°. Therefore we can use the lengths of $\overline{AC}$ and $\overline{BC}$ to find the exact value of the tangent of a 60° angle.

$$\tan 60° = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AC} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

But how can we find the constant value of the tangent ratio when the right triangle has an angle of 40° or 76°? Since we want to work with the value of this ratio for any right triangle, no matter what the measures of the acute angles may be, mathematicians have compiled tables of the tangent values for angles with measures from 0° to 90°. Also, a calculator has the ability to display the value of this ratio for any angle. We will use a calculator to determine these values.

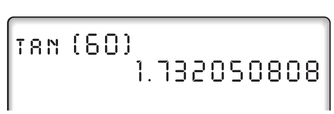
The measure of an angle can be given in degrees or in radians. In this book, we will always express the measure of an angle in degrees. A graphing calculator can use either radians or degrees. To place the calculator in degree mode, press **MODE**, then use the down arrow and the right arrow keys to highlight Deg. Press **ENTER** and **2nd** **QUIT**. Your calculator will be in degree mode each time you turn it on.

CASE I *Given an angle measure, find the tangent ratio.*

We saw that tan 60° is equal to $\sqrt{3}$. The calculator will display this value as an approximate decimal. To find tan 60°, enter the sequence of keys shown below.

ENTER: **TAN** 60 **)** **ENTER**

DISPLAY:



The value given in the calculator display is the rational approximation of $\sqrt{3}$, the value of tan 60° that we found using the ratio of the lengths of the legs of a right triangle with a 60° angle. Therefore, to the *nearest ten-thousandth*,

$$\tan 60° = 1.7321.$$

CASE 2 *Given a tangent ratio, find the angle measure.*

The value of the tangent ratio is different for each different angle measure from 0° to 90°. Therefore, if we know the value of the tangent ratio, we can find the measure of the acute angle that has this tangent ratio. The calculator key used to do this is labeled TAN⁻¹ and is accessed by first pressing 2nd. We can think of $\tan^{-1}$ as "the angle whose tangent is." Therefore, $\tan^{-1}(0.9004)$ can be read as "the angle whose tangent is 0.9004."

To find the measure of $\angle A$ from the calculator, we use the following sequences of keys.

ENTER: 2nd TAN⁻¹ 0.9004) ENTER

DISPLAY:
```
TAN-1(0.9004)
      41.99987203
```

The measure of $\angle A$ to the *nearest degree* is 42°.

EXAMPLE 1

In $\triangle ABC$, $\angle C$ is a right angle, $BC = 3$, $AC = 4$, and $AB = 5$.

a. Find:

(1) $\tan A$

(2) $\tan B$

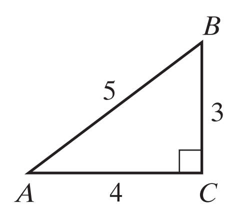
(3) $m\angle A$ to the *nearest degree*

(4) $m\angle B$ to the *nearest degree*

b. Show that the acute angles of the triangle are complementary.

Solution **a.** (1) $\tan A = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AC} = \frac{3}{4}$ *Answer*

(2) $\tan B = \frac{\text{opp}}{\text{adj}} = \frac{AC}{BC} = \frac{4}{3}$ *Answer*



Use a calculator to find the measures of $\angle A$ and $\angle B$.

(3) ENTER: 2nd TAN⁻¹ 3 ÷ 4) ENTER

DISPLAY:
```
TAN-1(3/4)
      36.86989765
```

To the nearest degree, $m\angle A = 37$. *Answer*

(4) ENTER: 2nd TAN⁻¹ 4 ÷ 3) ENTER

DISPLAY:
```
TAN⁻¹(4/3)
         53.13010235
```

To the nearest degree, m∠B = 53. *Answer*

b. m∠A + m∠B = 36.869889765 + 53.13010235 = 90.000000. Therefore, the acute angles of △ABC are complementary. *Answer*

Note: In a right triangle, the tangents of the two acute angles are reciprocals.

EXERCISES

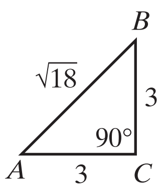
Writing About Mathematics

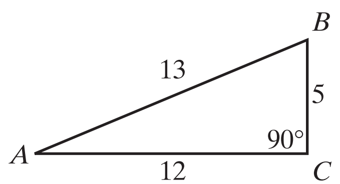
1. Explain why the tangent of a 45° angle is 1.

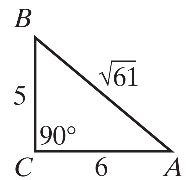
2. Use one of the right triangles formed by drawing an altitude of an equilateral triangle to find tan 30°. Express the answer that you find to the nearest hundred-thousandth and compare this result to the valued obtained from a calculator.

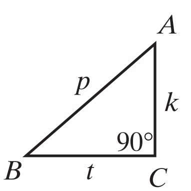
Developing Skills

In 3–6, find: **a.** tan A **b.** tan B

3. 

4. 

5. 

6. 

7. In △ABC, m∠C = 90, AC = 6, and AB = 10. Find tan A.

8. In △RST, m∠T = 90, RS = 13, and ST = 12. Find tan S.

In 9–16, use a calculator to find each of the following to the *nearest ten-thousandth*:

9. tan 10° **10.** tan 25° **11.** tan 70° **12.** tan 55°

13. tan 1° **14.** tan 89° **15.** tan 36° **16.** tan 67°

In 17–28, in each of the following, use a calculator to find the measure of ∠A to the *nearest degree*.

17. tan A = 0.0875 **18.** tan A = 0.3640 **19.** tan A = 0.5543

20. tan A = 1.0000 **21.** tan A = 2.0503 **22.** tan A = 3.0777

23. tan A = 0.3754 **24.** tan A = 0.7654 **25.** tan A = 1.8000

26. tan A = 0.3500 **27.** tan A = 0.1450 **28.** tan A = 2.9850

29. Does the tangent of an angle increase or decrease as the degree measure of the angle increases from 1° to 89°?

30. a. Use a calculator to find tan 20° and tan 40° to the *nearest ten-thousandth*.

b. Is the tangent of the angle doubled when the measure of the angle is doubled?

Applying Skills

31. In $\triangle ABC$, m$\angle C = 90$, $AC = 6$, and $BC = 6$.

a. Find tan A.

b. Find the measure of $\angle A$.

32. In $\triangle ABC$, m$\angle C = 90$, $BC = 4$, and $AC = 9$.

a. Find tan A.

b. Find the measure of $\angle A$ to the *nearest degree*.

c. Find tan B.

d. Find the measure of $\angle B$ to the *nearest degree*.

33. In rectangle $ABCD$, $AB = 10$ and $BC = 5$.

a. Find tan $\angle CAB$.

b. Find the measure of $\angle CAB$ to the *nearest degree*.

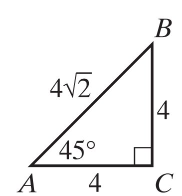
c. Find tan $\angle CAD$.

d. Find the measure of $\angle CAD$ to the *nearest degree*.

34. In $\triangle ABC$, $\angle C$ is a right angle, m$\angle A = 45$, $AC = 4$, $BC = 4$, and $AB = 4\sqrt{2}$.

a. Using the given lengths, write the ratio for tan A.

b. Use a calculator to find tan 45°.

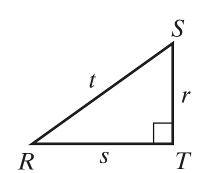


35. In $\triangle RST$, $\angle T$ is a right angle and r, s, and t are lengths of sides. Using these lengths:

a. Write the ratio for tan R.

b. Write the ratio for tan S.

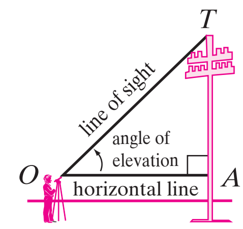
c. Use parts **a** and **b** to find the numerical value of the product (tan R)(tan S).



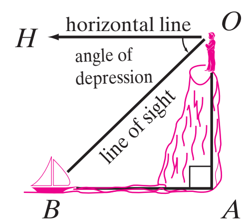
8-3 APPLICATIONS OF THE TANGENT RATIO

The tangent ratio is often used to make indirect measurements when the measures of a leg and an acute angle of a right triangle are known.

Angle of Elevation and Angle of Depression



When a telescope or some similar instrument is used to sight the top of a telephone pole, the instrument is elevated (tilted upward) from a horizontal position. Here, $\overleftrightarrow{OT}$ is the line of sight and $\overleftrightarrow{OA}$ is the horizontal line. The **angle of elevation** is the angle determined by the rays that are parts of the horizontal line and the line of sight when looking upward. Here, $\angle TOA$ is the angle of elevation.



When an instrument is used to sight a boat from a cliff, the instrument is depressed (tilted downward) from a horizontal position. Here, $\overleftrightarrow{OB}$ is the line of sight and $\overleftrightarrow{OH}$ is the horizontal line. The **angle of depression** is the angle determined by the rays that are parts of the horizontal line and of the line of sight when looking downward. Here, $\angle HOB$ is the angle of depression.

Note that, if $\overleftrightarrow{BA}$ is a horizontal line and $\overleftrightarrow{BO}$ is the line of sight from the boat to the top of the cliff, $\angle ABO$ is called the angle of elevation from the boat to the top of the cliff. Since $\overleftrightarrow{HO} \parallel \overleftrightarrow{BA}$ and $\overleftrightarrow{OB}$ is a transversal, alternate interior angles are congruent, namely, $\angle HOB \cong \angle ABO$. Thus, the angle of elevation measured from B to O is congruent to the angle of depression measured from O to B.

Using the Tangent Ratio to Solve Problems

Procedure

To solve a problem by using the tangent ratio:

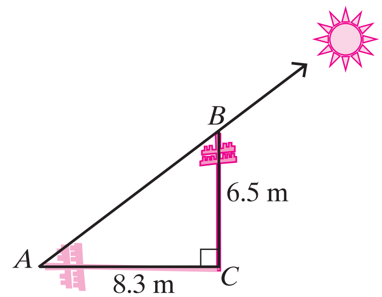
1. For the given problem, make a diagram that includes a right triangle. Label the known measures of the sides and angles. Identify the unknown quantity by a variable.
2. If for the right triangle either (1) the lengths of two legs or (2) the length of one leg and the measure of one acute angle are known, write a formula for the tangent of an acute angle.
3. Substitute known values in the formula and solve the resulting equation for the unknown value.

EXAMPLE 1

Find to the *nearest degree* the measure of the angle of elevation of the sun when a vertical pole 6.5 meters high casts a shadow 8.3 meters long.

Solution The angle of elevation of the sun is the same as $\angle A$, the angle of elevation to the top of the pole from A, the tip of the shadow. Since the vertical pole and the shadow are the legs of a right triangle opposite and adjacent to $\angle A$, use the tangent ratio.

$$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AC} = \frac{6.5}{8.3}$$



ENTER: **2nd** **TAN⁻¹** 6.5 **÷** 8.3 **)** **ENTER**

DISPLAY:

```
TAN⁻¹(6.5/8.3)
          38.0656511
```

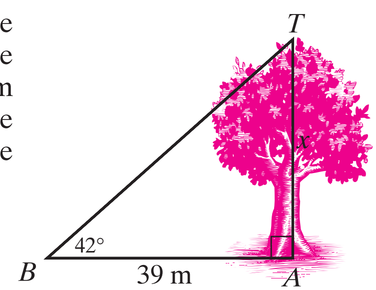
Answer To the *nearest degree*, the measure of the angle of elevation of the sun is 38°.

EXAMPLE 2

At a point on the ground 39 meters from the foot of a tree, the measure of the angle of elevation of the top of the tree is 42°. Find the height of the tree to the *nearest meter*.

Solution Let T be the top of the tree, A be the foot of the tree, and B be the point on the ground 39 meters from A. Draw $\triangle ABT$, and label the diagram: $m\angle B = 42$, $AB = 39$.

Let x = height of tree (AT). The height of the tree is the length of the perpendicular from the top of the tree to the ground. Since the problem involves the measure of an acute angle and the measures of the legs of a right triangle, use the tangent ratio:



$$\tan B = \frac{\text{opp}}{\text{adj}}$$

$$\tan B = \frac{AT}{BA}$$

$$\tan 42° = \frac{x}{39} \qquad \text{Substitute the given values.}$$

$$x = 39 \tan 42° \qquad \text{Solve for } x.$$

Use a calculator for the computation: $x = 35.11575773$

Answer To the *nearest meter*, the height of the tree is 35 meters.

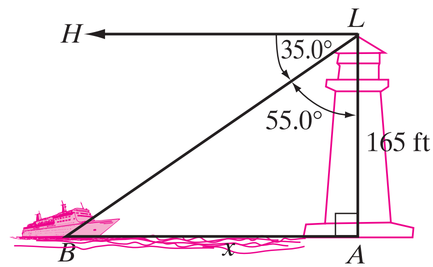
EXAMPLE 3

From the top of a lighthouse 165 feet above sea level, the measure of the angle of depression of a boat at sea is 35.0°. Find to the *nearest foot* the distance from the boat to the foot of the lighthouse.

Solution Let L be the top of the lighthouse, LA be the length of the perpendicular from L to sea level, and B be the position of the boat. Draw right triangle ABL, and draw $\overrightarrow{LH}$, the horizontal line through L.

Since $\angle HLB$ is the angle of depression, $m\angle HLB = 35.0$, $m\angle LBA = 35.0$, and $m\angle BLA = 90 - 35.0 = 55.0$.

Let x = distance from the boat to the foot of the lighthouse (BA).



METHOD 1

Using $\angle BLA$, $\overline{BA}$ is the opposite side and $\overline{LA}$ is the adjacent side. Form the tangent ratio:

$$\tan \angle BLA = \frac{BA}{LA}$$

$$\tan 55.0° = \frac{x}{165}$$

$$x = 165 \tan 55.0°$$

Use a calculator to perform the computation. The display will read 235.644421.

METHOD 2

Using $\angle LBA$, $\overline{LA}$ is the opposite side and $\overline{BA}$ is the adjacent side. Form the tangent ratio:

$$\tan \angle LBA = \frac{LA}{BA}$$

$$\tan 35.0° = \frac{165}{x}$$

$$x \tan 35.0° = 165$$

$$x = \frac{165}{\tan 35.0°}$$

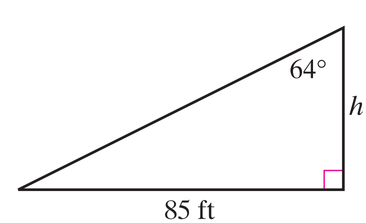
Use a calculator to perform the computation. The display will read 235.644421.

Answer To the *nearest foot*, the boat is 236 feet from the foot of the lighthouse.

EXERCISES

Writing About Mathematics

1. Zack is solving a problem in which the measure of the angle of depression from the top of a building to a point 85 feet from the foot of the building is 64°. To find the height of the building, Zack draws the diagram shown at the right. Explain why Zack's diagram is incorrect.

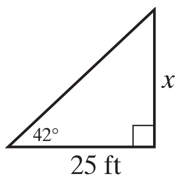


2. Explain why the angle of elevation from point A to point B is always congruent to the angle of depression from point B to point A.

Developing Skills

In 3–11, in each given triangle, find the length of the side marked x to the *nearest foot* or the measure of the angle marked x to the *nearest degree*.

3.



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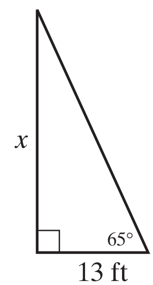

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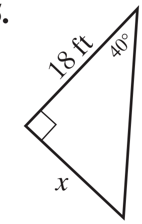

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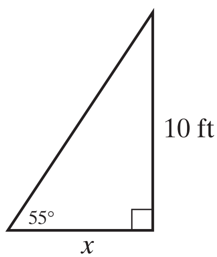

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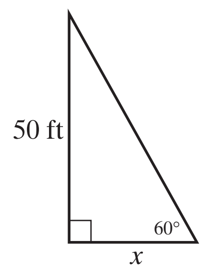

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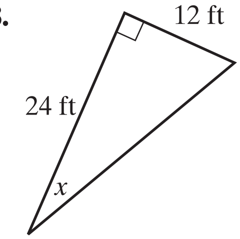

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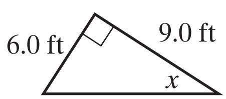

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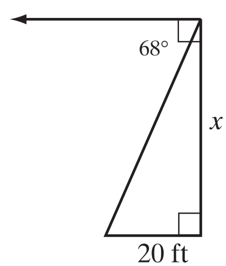

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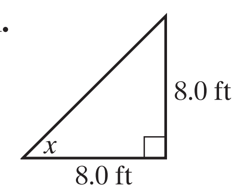


Applying Skills

12. At a point on the ground 52 meters from the foot of a tree, the measure of the angle of elevation of the top of the tree is 48°. Find the height of the tree to the *nearest meter*.

13. A ladder is leaning against a wall. The foot of the ladder is 6.25 feet from the wall. The ladder makes an angle of 74.5° with the level ground. How high on the wall does the ladder reach? Round the answer to the *nearest tenth of a foot*.

14. From a point, A, on the ground that is 938 feet from the foot, C, of the Empire State Building, the angle of elevation of the top, B, of the building has a measure of 57.5°. Find the height of the building to the *nearest ten feet*.

15. Find to the *nearest meter* the height of a building if its shadow is 18 meters long when the angle of elevation of the sun has a measure of 38°.

16. From the top of a lighthouse 50.0 meters high, the angle of depression of a boat out at sea has a measure of 15.0°. Find, to the *nearest meter*, the distance from the boat to the foot of the lighthouse, which is at sea level.

17. From the top of a school 61 feet high, the measure of the angle of depression to the road in front of the school is 38°. Find to the *nearest foot* the distance from the road to the school.

18. Find to the *nearest degree* the measure of the angle of elevation of the sun when a student 170 centimeters tall casts a shadow 170 centimeters long.

19. Find to the *nearest degree* the measure of the angle of elevation of the sun when a woman 150 centimeters tall casts a shadow 43 centimeters long.

20. A ladder leans against a building. The top of the ladder reaches a point on the building that is 18 feet above the ground. The foot of the ladder is 7.0 feet from the building. Find to the *nearest degree* the measure of the angle that the ladder makes with the level ground.

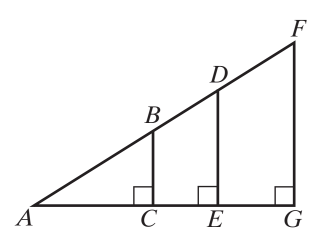
21. In any rhombus, the diagonals are perpendicular to each other and bisect each other. In rhombus $ABCD$, diagonals $\overline{AC}$ and $\overline{BD}$ meet at M. If $BD = 14$ and $AC = 20$, find the measure of each angle to the *nearest degree*.

a. $m\angle BCM$ **b.** $m\angle MBC$ **c.** $m\angle ABC$ **d.** $m\angle BCD$

8-4 THE SINE AND COSINE RATIOS

Since the tangent is the ratio of the lengths of the two legs of a right triangle, it is not directly useful in solving problems in which the hypotenuse is involved. In trigonometry of the right triangle, two ratios that involve the hypotenuse are called the **sine of an angle** and the **cosine of an angle**.

As in our discussion of the tangent ratio, we recognize that the figure at the right shows three similar triangles. Therefore, the ratios of corresponding sides are equal.



The Sine Ratio

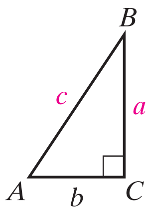
From the figure, we see that

$$\frac{BC}{AB} = \frac{DE}{AD} = \frac{FG}{AF} = \text{a constant}$$

This ratio is called the **sine of $\angle A$**.

DEFINITION

The **sine of an acute angle of a right triangle** is the ratio of the length of the side opposite the acute angle to the length of the hypotenuse.



In right triangle ABC, with $m\angle C = 90$, the definition of the *sine of* $\angle A$ is:

$$\textbf{sine } A = \frac{\text{length of side } opposite\ \angle A}{\text{length of } hypotenuse} = \frac{BC}{AB} = \frac{a}{c}$$

By using "sin A" as an abbreviation for sine A, "opp" as an abbreviation for the length of the leg opposite $\angle A$, and "hyp" as an abbreviation for the length of the hypotenuse, we can shorten the way we write the definition of sine A as follows:

$$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{BC}{AB} = \frac{a}{c}$$

The Cosine Ratio

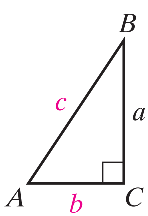
From the preceding figure on page 317, which shows similar triangles, $\triangle ABC$, $\triangle ADE$, and $\triangle AFG$, we see that

$$\frac{AC}{AB} = \frac{AE}{AD} = \frac{AG}{AF} = \text{a constant.}$$

This ratio is called the **cosine of $\angle A$**.

DEFINITION

The **cosine of an acute angle of a right triangle** is the ratio of the length of the side adjacent to the acute angle to the length of the hypotenuse.



In right triangle ABC, with $m\angle C = 90$, the definition of the *cosine of* $\angle A$ is:

$$\textbf{cosine } A = \frac{\text{length of side } adjacent \text{ to } \angle A}{\text{length of } hypotenuse} = \frac{AC}{AB} = \frac{b}{c}$$

By using "cos A" as an abbreviation for cosine A, "adj" as an abbreviation for the length of the leg adjacent to $\angle A$, and "hyp" as an abbreviation for the length of the hypotenuse, we can shorten the way we write the definition of cosine A as follows:

$$\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{AC}{AB} = \frac{b}{c}$$

Finding Sine and Cosine Ratios on a Calculator

CASE 1 *Given an angle measure, find the sine or cosine ratio.*

On a calculator we use the keys labeled **SIN** and **COS** to display the values of the sine and cosine of an angle. The sequence of keys that a calculator requires for tangent will be the same as the sequence for sine or cosine.

For example, to find sin 50° and cos 50°, we use the following:

ENTER: **SIN** 50 **)** **ENTER**

DISPLAY:
```
sin(50)
            .7660444431
```

ENTER: **COS** 50 **)** **ENTER**

DISPLAY:
```
cos(50)
            .6427876097
```

CASE 2 *Given a sine or cosine ratio, find the angle measure.*

A calculator will also find the measure of $\angle A$ when sin A or cos A is given. To do this we use the keys labeled **SIN⁻¹** and **COS⁻¹**. These are the second functions of **SIN** and **COS** and are accessed by first pressing **2nd**. We can think of the meaning of $\sin^{-1}$ as "the angle whose sine is." Therefore, if sin $A = 0.2588$, then $\sin^{-1}(0.2588)$ can be read as "the angle whose sine is 0.2588."

To find the measure of $\angle A$ from the calculator, we use the following sequences of keys:

ENTER: **2nd** **SIN⁻¹** 0.2588 **)** **ENTER**

DISPLAY:
```
sin⁻¹(0.2588)
            14.99887031
```

The measure of $\angle A$ to the *nearest degree* is 15°.

EXAMPLE 1

In $\triangle ABC$, $\angle C$ is a right angle, $BC = 7$, $AC = 24$, and $AB = 25$. Find:

a. $\sin A$

b. $\cos A$

c. $\sin B$

d. $\cos B$

e. $m\angle B$, to the *nearest degree*

Solution

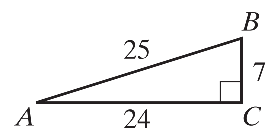
Answers

a. $\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{BC}{AB} = \frac{7}{25}$

b. $\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{AC}{AB} = \frac{24}{25}$

c. $\sin B = \frac{\text{opp}}{\text{hyp}} = \frac{AC}{AB} = \frac{24}{25}$

d. $\cos B = \frac{\text{adj}}{\text{hyp}} = \frac{BC}{AB} = \frac{7}{25}$



e. Use a calculator. Start with the ratio in part **c** and use SIN^{-1} or start with the ratio in part **d** and use COS^{-1}.

METHOD 1

$\sin B = \frac{24}{25}$

ENTER: 2nd SIN^{-1} 24 ÷ 25) ENTER

DISPLAY:

```
sin⁻¹(24/25)
        73.73979529
```

METHOD 2

$\cos B = \frac{7}{25}$

ENTER: 2nd COS^{-1} 7 ÷ 25) ENTER

DISPLAY:

```
cos⁻¹(7/25)
        73.73979529
```

$m\angle B = 74$ to the nearest degree. *Answer*

EXERCISES

Writing About Mathematics

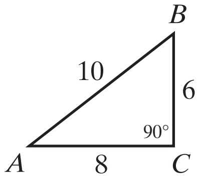
1. If $\angle A$ and $\angle B$ are the acute angles of right triangle ABC, show that $\sin A = \cos B$.

2. If $\angle A$ is an acute angle of right triangle ABC, explain why it is always true that $\sin A < 1$ and $\cos A < 1$.

Developing Skills

In 3–6, find: **a.** sin A **b.** cos A **c.** sin B **d.** cos B

3.



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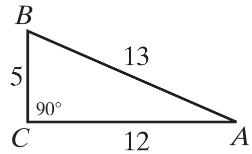

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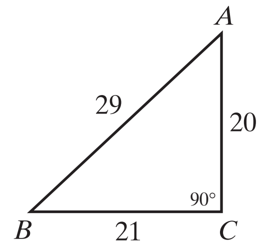

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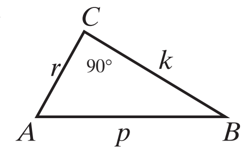


7. In $\triangle ABC$, m$\angle C = 90$, $AC = 4$, and $BC = 3$. Find sin A.

8. In $\triangle RST$, m$\angle S = 90$, $RS = 5$, and $ST = 12$. Find cos T.

In 9–20, for each of the following, use a calculator to find the trigonometric function value to the *nearest ten-thousandth*.

9. sin 18° **10.** sin 42° **11.** sin 58° **12.** sin 76°

13. sin 1° **14.** sin 89° **15.** cos 21° **16.** cos 35°

17. cos 40° **18.** cos 59° **19.** cos 74° **20.** cos 88°

In 21–38, for each of the following, use a calculator to find the measure of $\angle A$ to the *nearest degree*.

21. sin $A = 0.1908$ **22.** sin $A = 0.8387$ **23.** sin $A = 0.3420$

24. cos $A = 0.9397$ **25.** cos $A = 0.0698$ **26.** cos $A = 0.8910$

27. sin $A = 0.8910$ **28.** sin $A = 0.9986$ **29.** cos $A = 0.9986$

30. sin $A = 0.1900$ **31.** cos $A = 0.9750$ **32.** sin $A = 0.8740$

33. cos $A = 0.8545$ **34.** sin $A = 0.5800$ **35.** cos $A = 0.5934$

36. cos $A = 0.2968$ **37.** sin $A = 0.1275$ **38.** cos $A = 0.8695$

39. **a.** Use a calculator to find sin 25° and sin 50°.

b. If the measure of an angle is doubled, is the sine of the angle also doubled?

40. **a.** Use a calculator to find cos 25° and cos 50°.

b. If the measure of an angle is doubled, is the cosine of the angle also doubled?

41. As an angle increases in measure from 1° to 89°:

a. Does the sine of the angle increase or decrease?

b. Does the cosine of the angle increase or decrease?

In 42 and 43, complete each sentence by replacing ? with a degree measure that makes the sentence true.

42. **a.** $\sin 70° = \cos$?
b. $\sin 23° = \cos$?
c. $\sin 38° = \cos$?
d. $\sin x° = \cos$?

43. **a.** $\cos 50° = \sin$?
b. $\cos 17° = \sin$?
c. $\cos 82° = \sin$?
d. $\cos x° = \sin$?

Applying Skills

44. In $\triangle ABC$, $m\angle C = 90$, $BC = 20$, and $BA = 40$.
a. Find $\sin A$. **b.** Find the measure of $\angle A$.

45. In $\triangle ABC$, $m\angle C = 90$, $AC = 40$, and $AB = 80$.
a. Find $\cos A$. **b.** Find the measure of $\angle A$.

46. In $\triangle ABC$, $\angle C$ is a right angle, $AC = 8$, $BC = 15$, and $AB = 17$. Find:
a. $\sin A$ **b.** $\cos A$ **c.** $\sin B$ **d.** $\cos B$
e. the measure of $\angle A$ to the *nearest degree*
f. the measure of $\angle B$ to the *nearest degree*

47. In $\triangle RST$, $m\angle T = 90$, $ST = 11$, $RT = 60$, and $RS = 61$. Find:
a. $\sin R$ **b.** $\cos R$ **c.** $\sin S$ **d.** $\cos S$
e. the measure of $\angle R$ to the *nearest degree*
f. the measure of $\angle S$ to the *nearest degree*

48. In $\triangle ABC$, $\angle C$ is a right angle, $AC = 1.0$, $BC = 2.4$, and $AB = 2.6$. Find:
a. $\sin A$ **b.** $\cos A$ **c.** $\sin B$ **d.** $\cos B$
e. the measure of $\angle A$ to the *nearest degree*.
f. the measure of $\angle B$ to the *nearest degree*.

49. In rectangle $ABCD$, $AB = 3.5$ and $CB = 1.2$. Find:
a. $\sin \angle ABD$ **b.** $\cos \angle ABD$ **c.** $\sin \angle CBD$ **d.** $\cos \angle CBD$
e. the measure of $\angle ABD$ to the *nearest degree*.
f. the measure of $\angle CBD$ to the *nearest degree*.

50. In right triangle ABC, $\angle C$ is the right angle, $BC = 1$, $AC = \sqrt{3}$ and $AB = 2$.
a. Using the given lengths, write the ratios for $\sin A$ and $\cos A$.
b. Use a calculator to find $\sin 30°$ and $\cos 30°$.
c. What differences, if any, exist between the answers to parts **a** and **b**?

51. In $\triangle ABC$, $m\angle C = 90$ and $\sin A = \cos A$. Find $m\angle A$.

8-5 APPLICATIONS OF THE SINE AND COSINE RATIOS

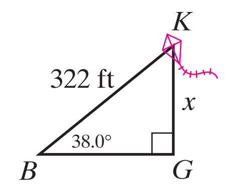
Since the sine and cosine ratios each have the length of the hypotenuse of a right triangle as the second term of the ratio, we can use these ratios to solve problems in the following cases:

1. We know the length of one leg and the measure of one acute angle and want to find the length of the hypotenuse.
2. We know the length of the hypotenuse and the measure of one acute angle and want to find the length of a leg.
3. We know the lengths of the hypotenuse and one leg and want to find the measure of an acute angle.

EXAMPLE 1

While flying a kite, Betty lets out 322 feet of string. When the string is secured to the ground, it makes an angle of 38.0° with the ground. To the *nearest foot*, what is the height of the kite above the ground? (Assume that the string is stretched so that it is straight.)

Solution Let K be the position of the kite in the air, B be the point on the ground at which the end of the string is secured, and G be the point on the ground directly below the kite, as shown in the diagram. The height of the kite is the length of the perpendicular from the ground to the kite. Therefore, $m\angle G = 90$, $m\angle B = 38.0$, and the length of the string $BK = 322$ feet.



Let $x = KG$, the height of the kite. We know the length of the hypotenuse and the measure of one acute angle and want to find the length of a leg, KG.

In **Method 1** below, since leg KG is opposite $\angle B$, we can use $\sin B = \frac{\text{opp}}{\text{hyp}}$.

In **Method 2** below, since leg KG is adjacent to $\angle K$, we can use $\cos K = \frac{\text{adj}}{\text{hyp}}$ with $m\angle K = 90 - 38.0 = 52.0$.

	METHOD 1	METHOD 2
Write the ratio:	$\sin B = \frac{KG}{BK}$	$\cos K = \frac{KG}{BK}$
Substitute the given values:	$\sin 38.0° = \frac{x}{322}$	$\cos 52.0° = \frac{x}{322}$
Solve for x:	$x = 322 \sin 38.0°$	$x = 322 \cos 52.0°$
Compute using a calculator:	$x = 198.242995$	$x = 198.242995$

Answer The height of the kite to the *nearest foot* is 198 feet.

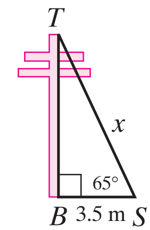
EXAMPLE 2

A wire reaches from the top of a pole to a stake in the ground 3.5 meters from the foot of the pole. The wire makes an angle of 65° with the ground. Find to the *nearest tenth of a meter* the length of the wire.

Solution In $\triangle BTS$, $\angle B$ is a right angle, $BS = 3.5$, $m\angle S = 65$, and $m\angle T = 90 - 65 = 25$.

Let $x = ST$, the length of the wire.

Since we know the length of one leg and the measure of one acute angle and want to find the length of the hypotenuse, we can use either the sine or the cosine ratio.



METHOD 1

$$\cos S = \frac{\text{adj}}{\text{hyp}} = \frac{BS}{ST}$$
$$\cos 65° = \frac{3.5}{x}$$
$$x \cos 65° = 3.5$$
$$x = \frac{3.5}{\cos 65°}$$
$$x = 8.281705541$$

METHOD 2

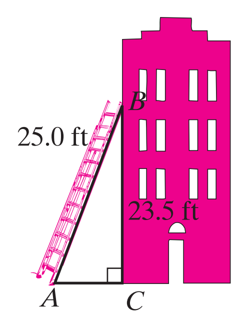
$$\sin T = \frac{\text{opp}}{\text{hyp}} = \frac{BS}{ST}$$
$$\sin 25° = \frac{3.5}{x}$$
$$x \sin 25° = 3.5$$
$$x = \frac{3.5}{\sin 25°}$$
$$x = 8.281705541$$

Answer The wire is 8.3 meters long to the *nearest tenth of a meter*.

EXAMPLE 3

A ladder 25 feet long leans against a building and reaches a point 23.5 feet above the ground. Find to the *nearest degree* the angle that the ladder makes with the ground.

Solution In right triangle ABC, AB, the length of the hypotenuse is 25 feet and BC, the side opposite $\angle A$, is 23.5 feet. Since the problem involves $\angle A$, $\overline{BC}$ (its opposite side), and $\overline{AB}$ (the hypotenuse), the sine ratio is used.



$$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{23.5}{25}$$

ENTER: **2nd** **SIN⁻¹** 23.5 **÷** 25 **)** **ENTER**

DISPLAY:

```
sin⁻¹(23.5/25.0)
          70.05155641
```

Answer To the nearest degree, the measure of the angle is 70°.

EXERCISES

Writing About Mathematics

1. Brittany said that for all acute angles, A, $(\tan A)(\cos A) = \sin A$. Do you agree with Brittany? Explain why or why not.

2. Pearl said that as the measure of an acute angle increases from 1° to 89°, the sine of the angle increases and the cosine of the angle decreases. Therefore, $\cos A$ is the reciprocal of $\sin A$. Do you agree with Pearl? Explain why or why not.

In 3–11, find to the *nearest centimeter* the length of the side marked x.

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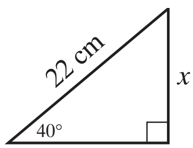

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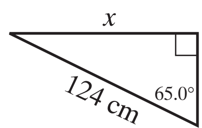

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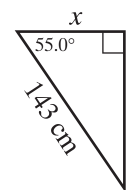

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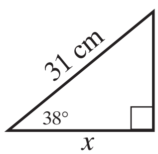

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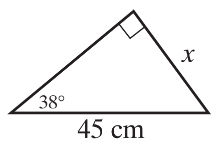

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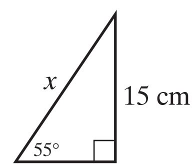

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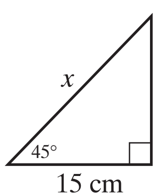

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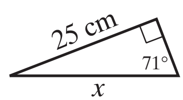

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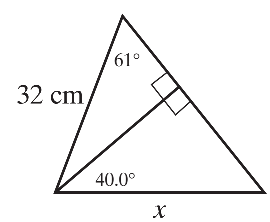
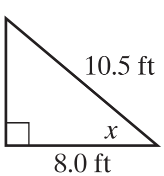


In 12–15, find to the *nearest degree* the measure of the angle marked x.

12.



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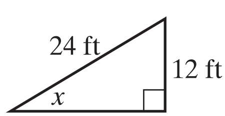

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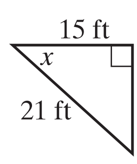

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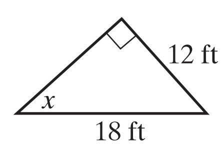


Applying Skills

16. A wooden beam 6.0 meters long leans against a wall and makes an angle of 71° with the ground. Find to the *nearest tenth of a meter* how high up the wall the beam reaches.

17. A boy flying a kite lets out 392 feet of string, which makes an angle of 52° with the ground. Assuming that the string is tied to the ground, find to the *nearest foot* how high the kite is above the ground.

18. A ladder that leans against a building makes an angle of 75° with the ground and reaches a point on the building 9.7 meters above the ground. Find to the *nearest meter* the length of the ladder.

19. From an airplane that is flying at an altitude of 3,500 feet, the angle of depression of an airport ground signal measures 27°. Find to the *nearest hundred feet* the distance between the airplane and the airport signal.

20. A 22-foot pole that is leaning against a wall reaches a point that is 18 feet above the ground. Find to the *nearest degree* the measure of the angle that the pole makes with the ground.

21. To reach the top of a hill that is 1.0 kilometer high, one must travel 8.0 kilometers up a straight road that leads to the top. Find to the *nearest degree* the measure of the angle that the road makes with the horizontal.

22. A 25-foot ladder leans against a building and makes an angle of 72° with the ground. Find to the *nearest foot* the distance between the foot of the ladder and the building.

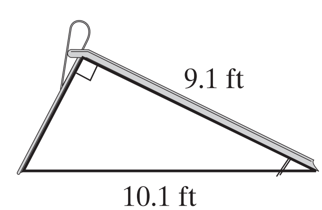
23. A wire 2.4 meters in length is attached from the top of a post to a stake in the ground. The measure of the angle that the wire makes with the ground is 35°. Find to the *nearest tenth of a meter* the distance from the stake to the foot of the post.

24. An airplane rises at an angle of 14° with the ground. Find to the *nearest hundred feet* the distance the airplane has flown when it has covered a horizontal distance of 1,500 feet.

25. A kite string makes an angle of 43° with the ground. The string is staked to a point 104 meters from a point on the ground directly below the kite. Find to the *nearest meter* the length of the kite string, which is stretched taut.

26. The top of a 43-foot ladder touches a point on the wall that is 36 feet above the ground. Find to the *nearest degree* the measure of the angle that the ladder makes with the wall.

27. In a park, a slide 9.1 feet long is perpendicular to the ladder to the top of the slide. The distance from the foot of the ladder to the bottom of the slide is 10.1 feet. Find to the *nearest degree* the measure of the angle that the slide makes with the horizontal.



28. A playground has the shape of an isosceles trapezoid $ABCD$. The length of the shorter base, $\overline{CD}$, is 185 feet. The length of each of the equal sides is 115 feet and $m\angle A = 65.0$.

a. Find DE, the length of the altitude from D, to the *nearest foot*.

b. Find AE, to the *nearest tenth of a foot*.

c. Find AB, to the *nearest foot*.

d. What is the area of the playground to the *nearest hundred square feet*?

e. What is the perimeter of the playground?

29. What is the area of a rhombus, to the *nearest ten square feet*, if the measure of one side is 43.7 centimeters and the measure of one angle is 78.0°?

30. A roofer wants to reach the roof of a house that is 21 feet above the ground. The measure of the steepest angle that a ladder can make with the house when it is placed directly under the roof is 27°. Find the length of the shortest ladder that can be used to reach the roof, to the *nearest foot*.

8-6 SOLVING PROBLEMS USING TRIGONOMETRIC RATIOS

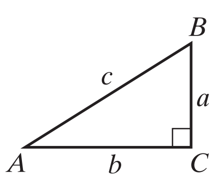
When the conditions of a problem can be modeled by a right triangle for which the measures of one side and an acute angle or of two sides are known, the trigonometric ratios can be used to find the measure of another side or of an acute angle.

Procedure

To solve a problem by using trigonometric ratios:

1. Draw the right triangle described in the problem.
2. Label the sides and angles with the given values.
3. Assign a variable to represent the measure to be determined.
4. Select the appropriate trigonometric ratio.
5. Substitute in the trigonometric ratio, and solve the resulting equation.

Given $\triangle ABC$ with $m\angle C = 90$:



$$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{BC}{AB} = \frac{a}{c} \qquad \sin B = \frac{\text{opp}}{\text{hyp}} = \frac{AC}{AB} = \frac{b}{c}$$

$$\cos A = \frac{\text{adj}}{\text{hyp}} = \frac{AC}{AB} = \frac{b}{c} \qquad \cos B = \frac{\text{adj}}{\text{hyp}} = \frac{BC}{AB} = \frac{a}{c}$$

$$\tan A = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AC} = \frac{a}{b} \qquad \tan B = \frac{\text{opp}}{\text{adj}} = \frac{AC}{BC} = \frac{b}{a}$$

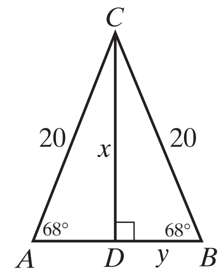
EXAMPLE 1

Given: In isosceles triangle ABC, $AC = CB = 20$ and $m\angle A = m\angle B = 68$. $\overline{CD}$ is an altitude.

Find:

a. Length of altitude $\overline{CD}$ to the *nearest tenth*.

b. Length of $\overline{AB}$ to the *nearest tenth*.



Solution **a.** In right $\triangle BDC$, $\sin B = \frac{CD}{CB}$

Let $x = CD$.

$$\sin 68° = \frac{x}{20}$$
$$x = 20 \sin 68°$$
$$= 18.54367709$$
$$= 18.5$$

b. Since the altitude drawn to the base of an isosceles triangle bisects the base, $AB = 2DB$. Therefore, find DB in $\triangle BDC$ and double it to find AB.

In right $\triangle BDC$, $\cos B = \frac{DB}{CB}$.

Let $y = DB$.

$$\cos 68° = \frac{y}{20}$$
$$y = 20 \cos 68° = 7.492131868$$
$$AB = 2y = 2(7.492131868) = 14.9843736$$

Answers **a.** $CD = 18.5$, to the *nearest tenth*. **b.** $AB = 15.0$, to the *nearest tenth*.

EXERCISES

Writing About Mathematics

1. If the measures of two sides of a right triangle are given, it is possible to find the measures of the third side and of the acute angles. Explain how you would find these measures.

2. If the measures of the acute angles of a right triangle are given, is it possible to find the measures of the sides? Explain why or why not.

Developing Skills

In 3–10: In each given right triangle, find to the *nearest foot* the length of the side marked x; or find to the *nearest degree* the measure of the angle marked x. Assume that each measure is given to the nearest foot or to the nearest degree.

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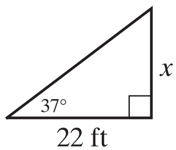

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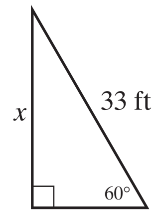

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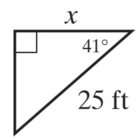

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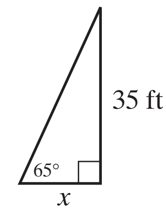

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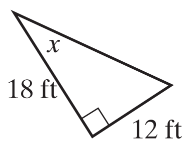

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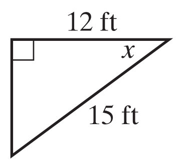

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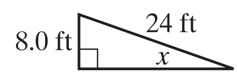

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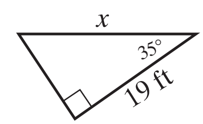


11. In $\triangle ABC$, $m\angle A = 42$, $AB = 14$, and $\overline{BD}$ is the altitude to $\overline{AC}$. Find BD to the *nearest tenth*.

12. In $\triangle ABC$, $\overline{AC} \cong \overline{BC}$, $m\angle A = 50$, and $AB = 30$. Find to the *nearest tenth* the length of the altitude from vertex C.

13. The legs of a right triangle measure 84 and 13. Find to the *nearest degree* the measure of the smallest angle of this triangle.

14. The length of hypotenuse $\overline{AB}$ of right triangle ABC is twice the length of leg $\overline{BC}$. Find the number of degrees in $\angle B$.

15. The longer side of a rectangle measures 10, and a diagonal makes an angle of 27° with this side. Find to the *nearest integer* the length of the shorter side.

16. In rectangle $ABCD$, diagonal $\overline{AC}$ measures 11 and side $\overline{AB}$ measures 7. Find to the *nearest degree* the measure of $\angle CAB$.

17. In right triangle ABC, $\overline{CD}$ is the altitude to hypotenuse $\overline{AB}$, $AB = 25$, and $AC = 20$. Find lengths AD, DB, and CD to the *nearest integer* and the measure of $\angle B$ to the *nearest degree*.

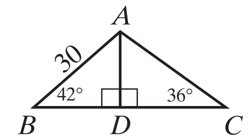
18. The lengths of the diagonals of a rhombus are 10 and 24.

a. Find the perimeter of the rhombus.

b. Find to the *nearest degree* the measure of the angle that the longer diagonal makes with a side of the rhombus.

19. The altitude to the hypotenuse of a right triangle ABC divides the hypotenuse into segments whose measures are 9 and 4. The measure of the altitude is 6. Find to the *nearest degree* the measure of the smaller acute angle of $\triangle ABC$.

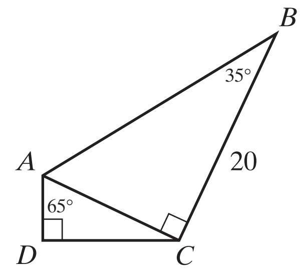
20. In $\triangle ABC$, $AB = 30$, $m\angle B = 42$, $m\angle C = 36$, and $\overline{AD}$ is an altitude.



a. Find to the *nearest integer* the length of $\overline{AD}$.

b. Using the result of part **a**, find to the *nearest integer* the length of $\overline{DC}$.

21. Angle D in quadrilateral $ABCD$ is a right angle, and diagonal $\overline{AC}$ is perpendicular to $\overline{BC}$, $BC = 20$, $m\angle B = 35$, and $m\angle DAC = 65$.



a. Find AC to the *nearest integer*.

b. Using the result of part **a**, find DC to the *nearest integer*.

22. The diagonals of a rectangle each measure 198 and intersect at an angle whose measure is 110°. Find to the *nearest integer* the length and width of the rectangle. *Hint:* The diagonals of a rectangle bisect each other.

23. In rhombus $ABCD$, the measure of diagonal $\overline{AC}$ is 80 and $m\angle BAC = 42$.

a. Find to the *nearest integer* the length of diagonal $\overline{BD}$.

b. Find to the *nearest integer* the length of a side of the rhombus.

24. In right triangle ABC, the length of hypotenuse $\overline{AB}$ is 100 and $m\angle A = 18$.

a. Find AC and BC to the *nearest integer*.

b. Show that the results of part **a** are approximately correct by using the relationship $(AB)^2 = (AC)^2 + (BC)^2$.

Applying Skills

25. Find to the *nearest meter* the height of a church spire that casts a shadow of 53.0 meters when the angle of elevation of the sun measures 68.0°.

26. From the top of a lighthouse 194 feet high, the angle of depression of a boat out at sea measures 34.0°. Find to the *nearest foot* the distance from the boat to the foot of the lighthouse.

27. A straight road to the top of a hill is 2,500 meters long and makes an angle of 12° with the horizontal. Find to the *nearest ten meters* the height of the hill.

28. A wire attached to the top of a pole reaches a stake in the ground 21 feet from the foot of the pole and makes an angle of 58° with the ground. Find to the *nearest foot* the length of the wire.

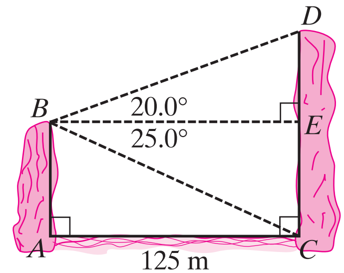
29. An airplane climbs at an angle of 11° with the ground. Find to the *nearest hundred feet* the distance the airplane has traveled when it has attained an altitude of 450 feet.

30. Find to the *nearest degree* the measure of the angle of elevation of the sun if a child 88 centimeters tall casts a shadow 180 centimeters long.

31. $\overline{AB}$ and $\overline{CD}$ represent cliffs on opposite sides of a river 125 meters wide. From B, the angle of elevation of D measures 20° and the angle of depression of C measures 25°. Find to the *nearest meter*:

a. the height of the cliff represented by $\overline{AB}$.

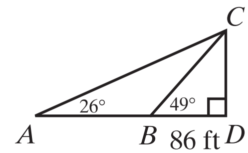
b. the height of the cliff represented by $\overline{CD}$.



32. Points A, B, and D are on level ground. $\overline{CD}$ represents the height of a building, $BD = 86$ feet, and $m\angle D = 90$. At B, the angle of elevation of the top of the building, $\angle CBD$, measures 49°. At A, the angle of elevation of the top of the building, $\angle CAD$, measures 26°.

a. Find the height of the building, CD, to the *nearest foot*.

b. Find AD to the *nearest foot*.



CHAPTER SUMMARY

The **Pythagorean Theorem** relates the lengths of the sides of a right triangle. If the lengths of the legs of a right triangle are a and b, and the length of the hypotenuse is c, then $c^2 = a^2 + b^2$.

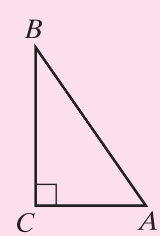
The **trigonometric functions** associate the measure of each acute angle A with a number that is the ratio of the measures of two sides of a right triangle. The three most commonly used trigonometric functions are **sine**, **cosine**, and **tangent**. In the application of trigonometry to the right triangle, these ratios are defined as follows:

$$\sin A = \frac{\text{opp}}{\text{hyp}} \qquad \cos A = \frac{\text{adj}}{\text{hyp}} \qquad \tan A = \frac{\text{opp}}{\text{adj}}$$

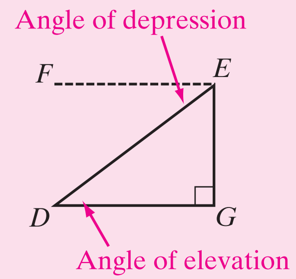
In right triangle ABC, with hypotenuse $\overline{AB}$

- $\overline{BC}$ is opposite $\angle A$ and adjacent to $\angle B$;
- $\overline{AC}$ is opposite $\angle B$ and adjacent to $\angle A$.

$$\sin A = \frac{BC}{AB} \qquad \cos B = \frac{BC}{AB}$$
$$\cos A = \frac{AC}{AB} \qquad \sin B = \frac{AC}{AB}$$
$$\tan A = \frac{BC}{AC} \qquad \tan B = \frac{AC}{BC}$$



An **angle of elevation**, $\angle GDE$ in the diagram, is an angle between a horizontal line and a line of sight to an object at a higher elevation. An **angle of depression**, $\angle FED$ in the diagram, is an angle between a horizontal line and a line of sight to an object at a lower elevation.



VOCABULARY

8-1 Trigonometry • Direct measurement • Indirect measurement • Pythagorean Theorem • Pythagorean triple

8-2 Opposite side • Adjacent side • Similar • Tangent of an acute angle of a right triangle

8-3 Angle of elevation • Angle of depression

8-4 Sine of an acute angle of a right triangle • Cosine of an acute angle of a right triangle

REVIEW EXERCISES

1. Talia's calculator is not functioning properly and does not give the correct value when she uses the TAN key. Assume that all other keys of the calculator are operating correctly.

a. Explain how Talia can find the measure of the leg $\overline{AC}$ of right triangle ABC when $BC = 4.5$ and $m\angle A = 43$.

b. Explain how Talia can use her calculator to find the tangent of any acute angle, given the measure of one side of a right triangle and the measure of an acute angle as in part **a**.

2. Jill made the following entry on her calculator:

ENTER: 2nd SIN^{-1} 1.5) ENTER

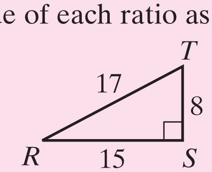
Explain why the calculator displayed an error message.

In 3–8, refer to $\triangle RST$ and express the value of each ratio as a fraction.

3. sin R **4.** tan T

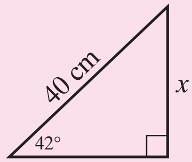
5. sin T **6.** cos R

7. cos T **8.** tan R

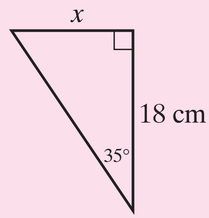


In 9–12: in each given triangle, find to the *nearest centimeter* the length of the side marked x. Assume that each given length is correct to the nearest centimeter.

9.



10.



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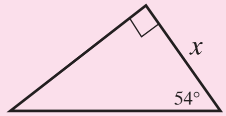
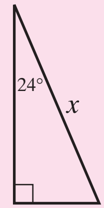


12.



13. If $\cos A = \sin 30°$ and $0° \leq A \leq 90°$, what is the measure of $\angle A$?

14. In right triangle ACB, $m\angle C = 90$, $m\angle A = 66$, and $AC = 100$. Find BC to the *nearest integer*.

15. In right triangle ABC, $m\angle C = 90$, $m\angle B = 28$, and $BC = 30$. Find AB to the *nearest integer*.

16. In $\triangle ABC$, $m\angle C = 90$, $\tan A = 0.7$, and $AC = 40$. Find BC.

17. In $\triangle ABC$, $m\angle C = 90$, $AB = 30$, and $BC = 15$. What is the measure, in degrees, of $\angle A$?

18. In $\triangle ABC$, $m\angle C = 90$, $BC = 5$, and $AC = 9$. Find to the *nearest degree* the measure of $\angle A$.

19. Find to the *nearest meter* the height of a building if its shadow is 42 meters long when the angle of elevation of the sun measures 42°.

20. A 5-foot wire attached to the top of a tent pole reaches a stake in the ground 3 feet from the foot of the pole. Find to the *nearest degree* the measure of the angle made by the wire with the ground.

21. While flying a kite, Doris let out 425 feet of string. Assuming that the string is stretched taut and makes an angle of 48° with the ground, find to the *nearest ten feet* how high the kite is.

22. A rectangular field $ABCD$ is crossed by a path from A to C. If $m\angle BAC = 62$ and $BC = 84$ yards, find to the *nearest yard*:

a. the width of the field, AB. **b.** the length of path, AC.

23. Find the length of a leg of an isosceles right triangle if the length of the hypotenuse is $\sqrt{72}$.

24. The measure of each of the base angles of an isosceles triangle is 15 degrees more than twice the measure of the vertex angle.

a. Find the measure of each angle of the triangle.

b. Find to the *nearest tenth of a centimeter* the measure of each of the equal sides of the triangle if the measure of the altitude to the base is 88.0 centimeters.

c. Find to the *nearest tenth of a centimeter* the measure of the base of the triangle.

d. Find the perimeter of the triangle.

e. Find the area of the triangle.

25. $ABCD$ is a rectangle with E a point on $\overline{BC}$. $AB = 12$, $BE = 5$, and $EC = 9$.

a. Find the perimeter of triangle AED.

b. Find the area of triangle AED.

c. Find the measure of $\angle CDE$.

d. Find the measure of $\angle BAE$.

e. Find the measure of $\angle AED$.

Exploration

A regular polygon with n sides can be divided into n congruent isosceles triangles.

a. Express, in terms of n, the measure of the vertex angle of one of the isosceles triangles.

b. Express, in terms of n, the measure of a base angle of one of the isosceles triangles.

c. Let s be the measure of a side of the regular polygon. Express, in terms of n and s, the measure of the altitude to the base of one of the isosceles triangles.

d. Express the area of one of the isosceles triangles in terms of n and s.

e. Write a formula for the area of a regular polygon in terms of the measure of a side, s, and the number of sides, n.

CUMULATIVE REVIEW — CHAPTERS 1–8

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. Which of the following does not represent a real number when $x = 3$?

(1) $\frac{3}{x}$ (2) $\frac{x}{x}$ (3) $\frac{x-3}{x}$ (4) $\frac{x}{x-3}$

2. The coordinates of one point on the x-axis are

(1) (1, 1) (2) (−1, −1) (3) (1, 0) (4) (0, 1)

3. The expression $-0.2a^2(10a^3 - 2a)$ is equivalent to

(1) $-2a^5 + 0.4a^3$ (3) $-2a^6 + 0.4a^2$
(2) $-20a^5 + 4a^3$ (4) $-20a^6 - 0.4a^3$

4. If $\frac{3}{4}x - 7 = \frac{1}{4}x + 3$, then x equals

(1) 20 (2) 10 (3) 5 (4) $\frac{5}{2}$

5. If $\sin A = 0.3751$, then, to the nearest degree, $m\angle A$ is

(1) 21 (2) 22 (3) 68 (4) 69

6. Which of the following statements is false?

(1) If a polygon is a square, then it is a parallelogram.
(2) If a polygon is a square, then it is a rhombus.
(3) If a polygon is a rectangle, then it is a parallelogram.
(4) If a polygon is a rectangle, then it is a rhombus.

7. If the measures of two legs of a right triangle are 7.0 feet and 8.0 feet, then, to the nearest tenth of a foot, the length of the hypotenuse is

(1) 10.6 (2) 15.0 (3) 41.2 (4) 48.9

8. The measure of the radius of a cylinder is 9.00 centimeters and the measure of its height is 24.00 centimeters. The surface area of the cylinder to the correct number of significant digits is

(1) 1,610 square centimeters (3) 1,860 square centimeters
(2) 1,620 square centimeters (4) 1,870 square centimeters

9. When $5b^2 + 2b$ is subtracted from $8b$ the difference is

(1) $6b - 5b^2$ (2) $5b^2 - 6b$ (3) $5b^2 - 10b$ (4) $-5b^2 - 6b$

10. When written in scientific notation, the fraction $\frac{(1.2 \times 10^{-4}) \times (3.5 \times 10^{8})}{8.4 \times 10^{-2}}$ equals

(1) 5.0×10^2 (2) 5.0×10^1 (3) 5.0×10^5 (4) 5.0×10^6

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. The vertices of pentagon $ABCDE$ are $A(2, -2)$, $B(7, -2)$, $C(7, 5)$, $D(0, 5)$, $E(-2, 0)$.

a. Draw pentagon $ABCDE$ on graph paper.

b. Find the area of the pentagon.

12. In $\triangle ABC$, $m\angle C = 90$, $m\angle B = 30$, $AC = 6x^2 - 4x$, and $AB = 2x$. Find the value of x.

Part III

Answer all questions in this part. Each correct answer will receive 3 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. Plank Road and Holt Road are perpendicular to each other. At a point 1.3 miles before the intersection of Plank and Holt, State Street crosses Plank at an angle of 57°. How far from the intersection of Plank and Holt will State Street intersect Holt? Write your answer to the nearest tenth of a mile.

14. Benny, Carlos, and Danny each play a different sport and have different career plans. Each of the four statements given below is true.

The boy who plays baseball plans to be an engineer.
Benny wants to be a lawyer.
Carlos plays soccer.
The boy who plans to be a doctor does not play basketball.

What are the career plans of each boy and what sport does he play?

Part IV

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. Bart wants to plant a garden around the base of a tree. To determine the amount of topsoil he will need to enrich his garden, he measured the circumference of the tree and found it to be 9.5 feet. His garden will be 2.0 feet wide, in the form of a ring around the tree. Find to the nearest square foot the surface area of the garden Bart intends to plant.

16. Samantha had a snapshot that is 3.75 inches wide and 6.5 inches high. She cut a strip off of the top of the snapshot so that an enlargement will fit into a frame that measures 5 inches by 8 inches. What were the dimensions of the strip that she cut off of the original snapshot?