

CHAPTER

12

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RATIO, PROPORTION, AND SIMILARITY

The relationship that we know as the Pythagorean Theorem was known by philosophers and mathematicians before the time of Pythagoras (c. 582–507 B.C.). The Indian mathematician Baudhāyana discovered the theorem more than 300 years before Pythagoras. The Egyptians made use of the 3-4-5 right triangle to determine a right angle. It may have been in Egypt, where Pythagoras studied, that he became aware of this relationship. Ancient sources agree that Pythagoras gave a proof of this theorem but no original documents exist from that time period.

Early Greek statements of this theorem did not use the algebraic form $c^2 = a^2 + b^2$ with which we are familiar. Proposition 47 in Book I of Euclid's *Elements* states the theorem as follows:

“In right angled triangles, the square on the side subtending the right angle is equal to the sum of the squares on the sides containing the right angle.”

Euclid's proof drew squares on the sides of the right triangle and proved that the area of the square drawn on the hypotenuse was equal to the sum of the areas of the squares drawn on the legs.

There exist today hundreds of proofs of this theorem.

12-1 RATIO AND PROPORTION

People often use a computer to share pictures with one another. At first the pictures may be shown on the computer screen as many small frames. These small pictures can be enlarged on the screen so that it is easier to see detail. Or a picture may be printed and then enlarged. Each picture, whether on the screen or printed, is *similar* to the original. When a picture is enlarged, the dimensions of each shape are in proportion to each other and with angle measure remaining the same. Shapes that are related in this way are said to be **similar**.

In this chapter we will review what we already know about ratio and proportion and apply those ideas to geometric figures.

The Meaning of Ratio

DEFINITION

The **ratio of two numbers**, a and b , where b is not zero, is the number $\frac{a}{b}$.

The ratio $\frac{a}{b}$ can also be written as $a : b$.

The two triangles $\triangle ABC$ and $\triangle DEF$ have the same shape but not the same size:

$AB = 20$ millimeters and $DE = 10$ millimeters. We can compare these lengths by means of a ratio, $\frac{20}{10}$ or $20 : 10$.

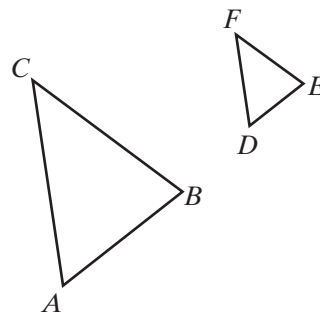
Since a ratio, like a fraction, is a comparison of two numbers by division, a ratio can be simplified by dividing each term of the ratio by a common factor. Therefore, the ratio of AB to DE can be written as $10 : 5$ or as $4 : 2$ or as $2 : 1$. A ratio is in simplest form when the terms of the ratio have no common factor greater than 1.

When the numbers represent lengths such as AB and DE , the lengths must be expressed in terms of the same unit of measure for the ratio to be meaningful.

For example, if AB had been given as 2 centimeters, it would have been necessary to change 2 centimeters to 20 millimeters before writing the ratio of AB to DE . Or we could have changed the length of DE , 10 millimeters, to 1 centimeter before writing the ratio of AB to DE as $2 : 1$.

- When using millimeters, the ratio $20 \text{ mm} : 10 \text{ mm} = 2 : 1$.
- When using centimeters, the ratio $2 \text{ cm} : 1 \text{ cm} = 2 : 1$.

A ratio can also be used to express the relationship among three or more numbers. For example, if the measures of the angles of a triangle are 45, 60, and 75, the ratio of these measures can be written as $45 : 60 : 75$ or, in lowest terms, $3 : 4 : 5$.



When we do not know the actual values of two or more measures that are in a given ratio, we use a variable factor to express these measures. For example, if the lengths of the sides of a triangle are in the ratio 3 : 3 : 4, we can let x be the greatest common factor of the measures of the sides. Then the measures of the sides may be expressed as $3x$, $3x$, and $4x$. If the perimeter of the triangle is 120 centimeters, this use of the variable x allows us to write and solve an equation.

$$3x + 3x + 4x = 120$$

$$10x = 120$$

$$x = 12$$

The measures of the sides of the triangle are $3(12)$, $3(12)$, and $4(12)$ or 36 centimeters, 36 centimeters, and 48 centimeters.

The Meaning of Proportion

Since the ratio 12 : 16 is equal to the ratio 3 : 4, we may write $\frac{12}{16} = \frac{3}{4}$. The equation $\frac{12}{16} = \frac{3}{4}$ is called a proportion. The proportion can also be written as $12 : 16 = 3 : 4$.

DEFINITION

A **proportion** is an equation that states that two ratios are equal.

The proportion $\frac{a}{b} = \frac{c}{d}$ can be written also as $a : b = c : d$. The four numbers a , b , c , and d are the terms of the proportion. The first and fourth terms, a and d , are the **extremes** of the proportion, and the second and third terms, b and c , are the **means**.

$$\begin{array}{c} \text{extremes} \\ \overbrace{a : b = c : d} \\ \underbrace{\hspace{1.5cm}} \\ \text{means} \end{array}$$

Theorem 12.1

In a proportion, the product of the means is equal to the product of the extremes.

Given $\frac{a}{b} = \frac{c}{d}$ with $b \neq 0$ and $d \neq 0$

Prove $ad = bc$

Proof We can give an algebraic proof of this theorem.

Statements	Reasons
1. $\frac{a}{b} = \frac{c}{d}$	1. Given.
2. $bd\left(\frac{a}{b}\right) = bd\left(\frac{c}{d}\right)$	2. Multiplication postulate.
3. $\frac{b}{b}(ad) = \frac{d}{d}(bc)$	3. Associative property of multiplication.
4. $1(ad) = 1(bc)$	4. A quantity may be substituted for its equal.
5. $ad = bc$	5. Multiplicative identity. 

Corollary 12.1a

In a proportion, the means may be interchanged.

Given $\frac{a}{b} = \frac{c}{d}$ with $b \neq 0, c \neq 0$, and $d \neq 0$

Prove $\frac{a}{c} = \frac{b}{d}$

Proof It is given that $\frac{a}{b} = \frac{c}{d}$ and that $c \neq 0$. Then $\frac{b}{c} \times \frac{a}{b} = \frac{b}{c} \times \frac{c}{d}$ by the multiplication postulate of equality. Therefore, $\frac{a}{c} = \frac{b}{d}$. 

Corollary 12.1b

In a proportion, the extremes may be interchanged.

Given $\frac{a}{b} = \frac{c}{d}$ with $a \neq 0, b \neq 0$, and $d \neq 0$

Prove $\frac{d}{b} = \frac{c}{a}$

Proof It is given that $\frac{a}{b} = \frac{c}{d}$ and that $a \neq 0$. Then $\frac{d}{a} \times \frac{a}{b} = \frac{d}{a} \times \frac{c}{d}$ by the multiplication postulate of equality. Therefore, $\frac{d}{b} = \frac{c}{a}$. 

These two corollaries tell us that:

If $3 : 5 = 12 : 20$, then $3 : 12 = 5 : 20$ and $20 : 5 = 12 : 3$.

Any two pairs of factors of the same number can be the means and the extremes of a proportion. For example, since $2(12) = 3(8)$, 2 and 12 can be the means of a proportion and 3 and 8 can be the extremes. We can write several proportions:

$$\frac{3}{2} = \frac{12}{8} \qquad \frac{8}{2} = \frac{12}{3} \qquad \frac{3}{12} = \frac{2}{8} \qquad \frac{8}{12} = \frac{2}{3}$$

The four proportions at the bottom of page 477 demonstrate the following corollary:

Corollary 12.1c

If the products of two pairs of factors are equal, the factors of one pair can be the means and the factors of the other the extremes of a proportion.

The Mean Proportional

DEFINITION

If the two means of a proportion are equal, either mean is called the **mean proportional** between the extremes of the proportion.

In the proportion $\frac{2}{6} = \frac{6}{18}$, 6 is the mean proportional between 2 and 18. The mean proportional is also called the **geometric mean**.

EXAMPLE 1

Solve for x : $\frac{27}{x+1} = \frac{9}{2}$

Solution Use that the product of the means is equal to the product of the extremes.

$$\begin{aligned}\frac{27}{x+1} &= \frac{9}{2} \\ 9(x+1) &= 27(2) \\ 9x + 9 &= 54 \\ 9x &= 45 \\ x &= 5\end{aligned}$$

Check

$$\begin{aligned}\frac{27}{x+1} &= \frac{9}{2} \\ \frac{27}{5+1} &\stackrel{?}{=} \frac{9}{2} && \text{Substitute 5 for } x. \\ \frac{27}{6} &\stackrel{?}{=} \frac{9}{2} && \text{Simplify.} \\ \frac{9}{2} &= \frac{9}{2} \quad \checkmark\end{aligned}$$

Answer $x = 5$

EXAMPLE 2

Find the mean proportional between 9 and 8.

Solution Let x represent the mean proportional.

$$\begin{aligned}\frac{9}{x} &= \frac{x}{8} \\ x^2 &= 72 \\ x &= \pm\sqrt{72} = \pm\sqrt{36}\sqrt{2} = \pm 6\sqrt{2}\end{aligned}$$

Note that there are two solutions, one positive and one negative.

Answer $\pm 6\sqrt{2}$

EXAMPLE 3

The measures of an exterior angle of a triangle and the adjacent interior angle are in the ratio 7 : 3. Find the measure of the exterior angle.

Solution An exterior angle and the adjacent interior angle are supplementary.

Let $7x$ = the measure of the exterior angle,

and $3x$ = the measure of the interior angle.

$$7x + 3x = 180$$

$$10x = 180$$

$$x = 18$$

$$7x = 126$$

Answer The measure of the exterior angle is 126° . 

Exercises**Writing About Mathematics**

1. Carter said that a proportion can be rewritten by using the means as extremes and the extremes as means. Do you agree with Carter? Explain why or why not.
2. Ethan said that the mean proportional will be a rational number only if the extremes are both perfect squares. Do you agree with Ethan? Explain why or why not.

Developing Skills

In 3–8, determine whether each pair of ratios can form a proportion.

3. $6 : 15, 4 : 10$

4. $8 : 7, 56 : 49$

5. $49 : 7, 1 : 7$

6. $10 : 15, 8 : 12$

7. $9 : 3, 16 : 4$

8. $3a : 5a, 12 : 20$ ($a \neq 0$)

In 9–11, use each set of numbers to form two proportions.

9. 30, 6, 5, 1

10. 18, 12, 6, 4

11. 3, 10, 15, 2

12. Find the exact value of the geometric mean between 10 and 40.

13. Find the exact value of the geometric mean between 6 and 18.

In 14–19, find the value of x in each proportion.

14. $4 : x = 10 : 15$

15. $\frac{9}{8} = \frac{x}{36}$

16. $\frac{12}{x+1} = \frac{8}{x}$

17. $12 : x = x : 75$

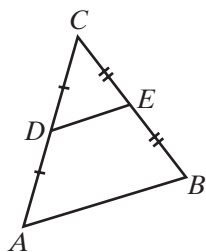
18. $3x : 15 = 20 : x$

19. $x + 3 : 6 = 4 : x - 2$

Applying Skills

20. B is a point on $\overline{ABC}$ such that $AB : BC = 4 : 7$. If $AC = 33$, find AB and BC .
21. A line segment 48 centimeters long is divided into two segments in the ratio $1 : 5$. Find the measures of the segments.
22. A line segment is divided into two segments that are in the ratio $3 : 5$. The measure of one segment is 12 centimeters longer than the measure of the other. Find the measure of each segment.
23. The measures of the sides of a triangle are in the ratio $5 : 6 : 7$. Find the measure of each side if the perimeter of the triangle is 72 inches.
24. Can the measures of the sides of a triangle be in the ratio $2 : 3 : 7$? Explain why or why not.
25. The length and width of a rectangle are in the ratio $5 : 8$. If the perimeter of the rectangle is 156 feet, what are the length and width of the rectangle?
26. The measures of two consecutive angles of a parallelogram are in the ratio $2 : 7$. Find the measure of each angle.

12-2 PROPORTIONS INVOLVING LINE SEGMENTS



The midpoint of any line segment divides the segment into two congruent parts. In $\triangle ABC$, let D be the midpoint of $\overline{AC}$ and E be the midpoint of $\overline{BC}$. Draw the midsegment, $\overline{DE}$.

The line segment joining the midpoints of $\triangle ABC$ forms a new triangle, $\triangle DEC$. What are the ratios of the sides of these triangles?

- D is the midpoint of $\overline{AC}$. Therefore, $DC = \frac{1}{2}AC$ and $\frac{DC}{AC} = \frac{1}{2}$.
- E is the midpoint of $\overline{BC}$. Therefore, $EC = \frac{1}{2}BC$ and $\frac{EC}{BC} = \frac{1}{2}$.

If we measure $\overline{AB}$ and $\overline{DE}$, it appears that $DE = \frac{1}{2}AB$ and $\frac{DE}{AB} = \frac{1}{2}$. It also appears that $\overline{AB} \parallel \overline{DE}$. We can prove these last two observations as a theorem called the **midsegment theorem**.

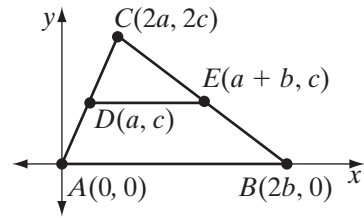
Theorem 12.2

A line segment joining the midpoints of two sides of a triangle is parallel to the third side and its length is one-half the length of the third side.

Given $\triangle ABC$, D is the midpoint of $\overline{AC}$, and E is the midpoint of $\overline{BC}$.

Prove $\overline{DE} \parallel \overline{AB}$ and $DE = \frac{1}{2}AB$

Proof We will use a coordinate proof for this theorem. The triangle can be placed at any convenient position. We will place A at the origin and B on the x -axis. Let the coordinates of the vertices of $\triangle ABC$ be $A(0, 0)$, $B(2b, 0)$, and $C(2a, 2c)$. Then:



- The coordinates of D are $\left(\frac{2a+0}{2}, \frac{2c+0}{2}\right) = (a, c)$.
- The coordinates of E are $\left(\frac{2a+2b}{2}, \frac{2c+0}{2}\right) = (a+b, c)$.
- The slope of $\overline{AB}$ is $\frac{0-0}{2b-0} = 0$. $\overline{AB}$ is a horizontal line segment.
- The slope of $\overline{DE}$ is $\frac{c-c}{a+b-a} = 0$. $\overline{DE}$ is a horizontal line segment.

Therefore, $\overline{AB}$ and $\overline{DE}$ are parallel line segments because horizontal line segments are parallel.

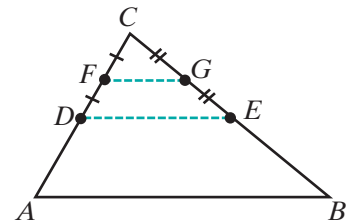
The length of a horizontal line segment is the absolute value of the difference of the x -coordinates of the endpoints.

$$\begin{aligned} AB &= |2b - 0| & DE &= |(a+b) - a| \\ &= |2b| & &= |b| \end{aligned} \quad \text{and}$$

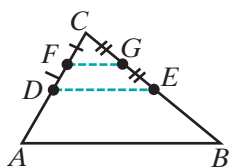
Therefore, $DE = \frac{1}{2}AB$. ■

Now that we know that our observations are correct, that $DE = \frac{1}{2}AB$ and that $\overline{AB} \parallel \overline{DE}$, we know that $\angle A \cong \angle EDC$ and $\angle B \cong \angle DEC$ because they are corresponding angles of parallel lines. We also know that $\angle C \cong \angle C$. Therefore, for $\triangle ABC$ and $\triangle DEC$, the corresponding angles are congruent and the ratios of the lengths of corresponding sides are equal.

Again, in $\triangle ABC$, let D be the midpoint of $\overline{AC}$ and E be the midpoint of $\overline{BC}$. Draw $\overline{DE}$. Now let F be the midpoint of $\overline{DC}$ and G be the midpoint of $\overline{EC}$. Draw $\overline{FG}$. We can derive the following information from the segments formed:



- $FC = \frac{1}{2}DC = \frac{1}{2}\left(\frac{1}{2}AC\right) = \frac{1}{4}AC$ or $\frac{FC}{AC} = \frac{1}{4}$
- $GC = \frac{1}{2}EC = \frac{1}{2}\left(\frac{1}{2}BC\right) = \frac{1}{4}BC$ or $\frac{GC}{BC} = \frac{1}{4}$
- $FG = \frac{1}{2}DE = \frac{1}{2}\left(\frac{1}{2}AB\right) = \frac{1}{4}AB$ or $\frac{FG}{AB} = \frac{1}{4}$ (by Theorem 12.2)
- Let $AC = 4x$. Then $AD = 2x$, $DC = 2x$, $DF = x$, and $FC = x$.



- Let $BC = 4y$. Then $BE = 2y$, $EC = 2y$, $EG = y$, and $GC = x$.
Also, $BG = 2y + y = 3y$.

- Therefore, $\frac{FC}{AF} = \frac{x}{3x} = \frac{1}{3}$ and $\frac{GC}{BG} = \frac{y}{3y} = \frac{1}{3}$.

Since $\frac{FC}{AF}$ and $\frac{GC}{BG}$ are each equal to $\frac{1}{3}$, $\frac{FC}{AF} = \frac{GC}{BG}$. We say that the points F and G divide $\overline{AC}$ and $\overline{BC}$ proportionally because these points separate the segments into parts whose ratios form a proportion.

DEFINITION

Two line segments are **divided proportionally** when the ratio of the lengths of the parts of one segment is equal to the ratio of the lengths of the parts of the other.

The points D and E also divide $\overline{AC}$ and $\overline{BC}$ proportionally because these points also separate the segments into parts whose ratios form a proportion.

$$\frac{AD}{DC} = \frac{2x}{2x} = 1 \quad \text{and} \quad \frac{BE}{EC} = \frac{2y}{2y} = 1$$

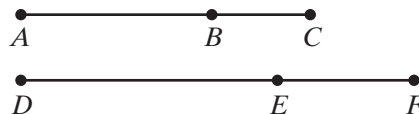
$$\text{Therefore, } \frac{AD}{DC} = \frac{BE}{EC}.$$

Theorem 12.3a

If two line segments are divided proportionally, then the ratio of the length of a part of one segment to the length of the whole is equal to the ratio of the corresponding lengths of the other segment.

Given $\overline{ABC}$ and $\overline{DEF}$ with $\frac{AB}{BC} = \frac{DE}{EF}$.

Prove $\frac{AB}{AC} = \frac{DE}{DF}$



Proof

Statements	Reasons
1. $\frac{AB}{BC} = \frac{DE}{EF}$	1. Given.
2. $(AB)(EF) = (BC)(DE)$	2. The product of the means equals the product of the extremes.
3. $(AB)(EF) + (AB)(DE) = (BC)(DE) + (AB)(DE)$	3. Addition postulate.
4. $(AB)(EF + DE) = (DE)(BC + AB)$	4. Distributive property.
5. $(AB)(DF) = (DE)(AC)$	5. Substitution postulate.
6. $\frac{AB}{AC} = \frac{DE}{DF}$	6. If the products of two pairs of factors are equal, one pair of factors can be the means and the other the extremes of a proportion. ■

Theorem 12.3b

If the ratio of the length of a part of one line segment to the length of the whole is equal to the ratio of the corresponding lengths of another line segment, then the two segments are divided proportionally.

The proof of this theorem is left to the student. (See exercise 21.) Theorems 12.3a and 12.3b can be written as a biconditional.

Theorem 12.3

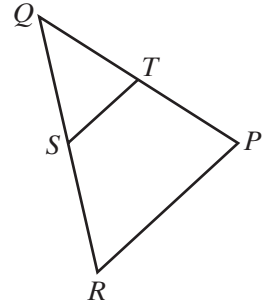
Two line segments are divided proportionally if and only if the ratio of the length of a part of one segment to the length of the whole is equal to the ratio of the corresponding lengths of the other segment.

EXAMPLE 1

In $\triangle PQR$, S is the midpoint of $\overline{RQ}$ and T is the midpoint of $\overline{PQ}$.

$$RP = 7x + 5 \quad ST = 4x - 2 \quad SR = 2x + 1 \quad PQ = 9x + 1$$

Find ST , RP , SR , RQ , PQ , and TQ .



Solution The length of the line joining the midpoints of two sides of a triangle is equal to one-half the length of the third side.

$$\begin{aligned} 4x - 2 &= \frac{1}{2}(7x + 5) \\ 2(4x - 2) &= 2\left(\frac{1}{2}\right)(7x + 5) \\ 8x - 4 &= 7x + 5 \\ x &= 9 \end{aligned}$$

$$\begin{aligned} ST &= 4(9) - 2 = 36 - 2 = 34 \\ RP &= 7(9) + 5 = 63 + 5 = 68 \\ SR &= 2(9) + 1 = 18 + 1 = 19 \\ RQ &= 2SR = 2(19) = 38 \\ PQ &= 9(9) + 1 = 81 + 1 = 82 \\ TQ &= \frac{1}{2}PQ = \frac{1}{2}(82) = 41 \end{aligned}$$

EXAMPLE 2

$\overline{ABC}$ and $\overline{DEF}$ are line segments. If $AB = 10$, $AC = 15$, $DE = 8$, and $DF = 12$, do B and E divide $\overline{ABC}$ and $\overline{DEF}$ proportionally?

Solution If $AB = 10$ and $AC = 15$, then:

$$\begin{aligned} BC &= 15 - 10 & AB:BC &= 10:5 \\ &= 5 & &= 2:1 \end{aligned}$$

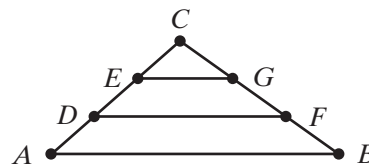
If $DE = 8$ and $DF = 12$, then:

$$\begin{aligned} EF &= 12 - 8 & DE:EF &= 8:4 \\ &= 4 & &= 2:1 \end{aligned}$$

Since the ratios of $AB:BC$ and $DE:EF$ are equal, B and E divide $\overline{ABC}$ and $\overline{DEF}$ proportionally.

EXAMPLE 3

In the diagram, $\overline{ADEC}$ and $\overline{BFGC}$ are two sides of $\triangle ABC$. If $AD = DE = EC$ and $BF = FG = GC$, prove that $EG = \frac{1}{2}DF$.



Solution Since D and E are on $\overline{ADEC}$ and $DE = EC$, E is the midpoint of $\overline{DEC}$.

Since F and G are on $\overline{BFGC}$ and $FG = GC$, G is the midpoint of $\overline{FGC}$.

In $\triangle DFC$, $\overline{EG}$ is the line segment joining the midpoints of two sides of the triangle. By Theorem 12.2, $\overline{EG}$, a line segment joining the midpoints of two sides of a triangle, is parallel to the third side, $\overline{DF}$, and its length is one-half the length of the third side. Therefore, $EG = \frac{1}{2}DF$. ■

Exercises

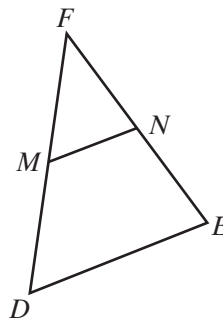
Writing About Mathematics

1. Explain why the midpoints of two line segments always divide those segments proportionally.
2. Points B , C , D , and E divide $\overline{ABCDEF}$ into five equal parts. Emily said that $AB : BF = 1 : 5$. Do you agree with Emily? Explain why or why not.

Developing Skills

In 3–10, M is the midpoint of $\overline{DF}$ and N is the midpoint of $\overline{EF}$.

3. Find DE if $MN = 9$.
4. Find MN if $DE = 17$.
5. Find DM if $DF = 24$.
6. Find NF if $EF = 10$.
7. Find $DM : DF$.
8. Find $DP : PF$ if P is the midpoint of $\overline{MF}$.
9. Find $m\angle FMN$ if $m\angle D = 76$.
10. Find $m\angle ENM$ if $m\angle E = 42$.



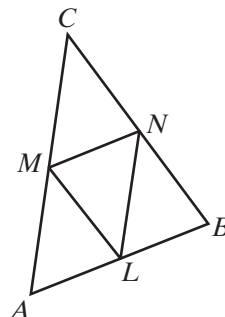
11. The length of the diagonal of a rectangle is 12 centimeters. What is the measure of a line segment that joins the midpoints of two consecutive sides of the rectangle?

In 12–15, the line segments $\overline{ABC}$ and $\overline{PQR}$ are divided proportionally by B and Q . $AB < BC$ and $PQ < QR$.

12. Find PQ when $AB = 15$, $BC = 25$, and $QR = 35$.
13. Find BC when $AB = 8$, $PQ = 20$, and $PR = 50$.
14. Find AC when $AB = 12$, $QR = 27$, and $BC = PQ$.
15. Find AB and BC when $AC = 21$, $PQ = 14$, and $QR = 35$.
16. Line segment $\overline{KLMN}$ is divided by L and M such that $KL : LM : MN = 2 : 4 : 3$. Find:
 - a. $KL : KN$
 - b. $LN : MN$
 - c. $LM : LN$
 - d. $KM : LN$
17. Line segment $\overline{ABC}$ is divided by B such that $AB : BC = 2 : 3$ and line segment $\overline{DEF}$ is divided by E such that $DE : EF = 2 : 3$. Show that $AB : AC = DE : DF$.

Applying Skills

18. The midpoint the sides of $\triangle ABC$ are L , M , and N .
 - a. Prove that quadrilateral $LMCN$ is a parallelogram.
 - b. If $AB = 12$, $BC = 9$, and $CA = 15$, what is the perimeter of $LMCN$?

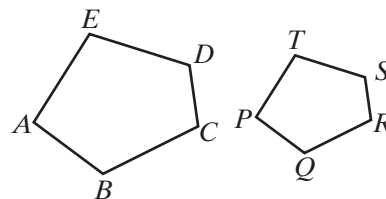


19. In right triangle ABC , the midpoint of the hypotenuse $\overline{AB}$ is M and the midpoints of the legs are P and Q . Prove that quadrilateral $PMQC$ is a rectangle.
20. In right triangle ABC , the midpoint of the hypotenuse $\overline{AB}$ is M , the midpoint of $\overline{BC}$ is P , and the midpoint of $\overline{CA}$ is Q . D is a point on $\overleftrightarrow{PM}$ such that $PM = MD$.
 - a. Prove that $QADM$ is a rectangle.
 - b. Prove that $\overline{CM} \cong \overline{AM}$.
 - c. Prove that M is equidistant from the vertices of $\triangle ABC$.
21. Prove Theorem 12.3b, “If the ratio of the length of a part of one line segment to the length of the whole is equal to the ratio of the corresponding lengths of another line segment, then the two segments are divided proportionally.”
22. The midpoints of the sides of quadrilateral $ABCD$ are M , N , P , and Q . Prove that quadrilateral $MNPQ$ is a parallelogram. (*Hint: Draw $\overline{AC}$.*)

12-3 SIMILAR POLYGONS

Two polygons that have the same shape but not the same size are called similar polygons. In the figure to the right, $ABCDE \sim PQRST$. The symbol $\sim$ is read “is similar to.”

These polygons have the same shape because their corresponding angles are congruent and the ratios of the lengths of their corresponding sides are equal.



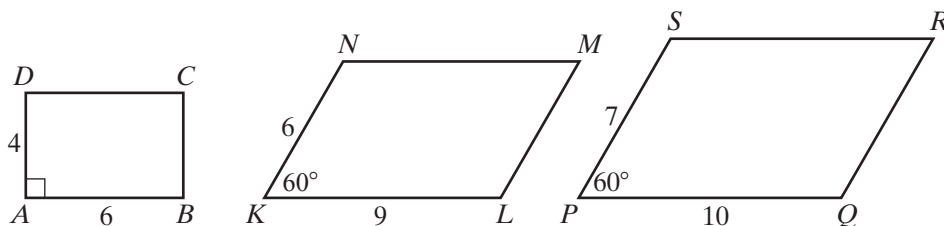
DEFINITION

Two polygons are similar if there is a one-to-one correspondence between their vertices such that:

1. All pairs of corresponding angles are congruent.
2. The ratios of the lengths of all pairs of corresponding sides are equal.

When the ratios of the lengths of the corresponding sides of two polygons are equal, as shown in the example above, we say that the corresponding sides of the two polygons are in proportion. The ratio of the lengths of corresponding sides of similar polygons is called the **ratio of similitude** of the polygons. The number represented by the ratio of similitude is called the **constant of proportionality**.

Both conditions mentioned in the definition must be true for polygons to be similar.



Rectangle $ABCD$ is *not* similar to parallelogram $KLMN$. The corresponding sides are in proportion, $\frac{4}{6} = \frac{6}{9}$, but the corresponding angles are not congruent.

Parallelogram $KLMN$ is *not* similar to parallelogram $PQRS$. The corresponding angles are congruent but the corresponding sides are not in proportion, $\frac{6}{9} \neq \frac{7}{10}$.

Recall that a mathematical definition is reversible:

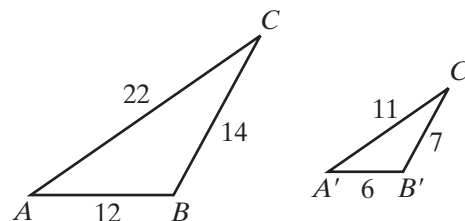
- If two polygons are similar, then their corresponding angles are congruent and their corresponding sides are in proportion.

and

- If two polygons have corresponding angles that are congruent and corresponding sides that are in proportion, then the polygons are similar.

Since triangles are polygons, the definition given for two similar polygons will apply also to two similar triangles.

In the figures to the right, $\triangle ABC \sim \triangle A'B'C'$. We can draw the following conclusions about the two triangles:



$$\begin{array}{lll} \angle A \cong \angle A' & \angle B \cong \angle B' & \angle C \cong \angle C' \\ AB:A'B' = 12:6 & BC:B'C' = 14:7 & CA:C'A' = 22:11 \\ = 2:1 & = 2:1 & = 2:1 \end{array}$$

The ratio of similitude for the triangles is 2 : 1.

Equivalence Relation of Similarity

The relation “is similar to” is true for polygons when their corresponding angles are congruent and their corresponding sides are in proportion. Thus, for a given set of triangles, we can test the following properties:

1. Reflexive property: $\triangle ABC \sim \triangle ABC$. (Here, the ratio of the lengths of corresponding sides is 1 : 1.)
2. Symmetric property: If $\triangle ABC \sim \triangle DEF$, then $\triangle DEF \sim \triangle ABC$.
3. Transitive property: If $\triangle ABC \sim \triangle DEF$, and $\triangle DEF \sim \triangle RST$, then $\triangle ABC \sim \triangle RST$.

These properties for any similar geometric figures can be stated as postulates.

Postulate 12.1

Any geometric figure is similar to itself. (Reflexive property)

Postulate 12.2

A similarity between two geometric figures may be expressed in either order. (Symmetric property)

Postulate 12.3

Two geometric figures similar to the same geometric figure are similar to each other. (Transitive property)

EXAMPLE 1

In right triangle ABC , $m\angle A = 67.4$, $AB = 13.0$, $BC = 12.0$, and $CA = 5.00$.

In right triangle DEF , $m\angle E = 22.6$, $DE = 19.5$, $EF = 18.0$, and $FD = 7.50$.

Prove that $\triangle ABC \sim \triangle DEF$.

Proof Triangles ABC and DEF are right triangles. The angles opposite the longest sides are right angles. Therefore, $m\angle C = 90$, $m\angle F = 90$, and $\angle C \cong \angle F$.

The acute angles of a right triangle are complementary. Therefore, $m\angle B = 90 - m\angle A = 90 - 67.4 = 22.6$, and $\angle B \cong \angle E$.

Similarly, $m\angle D = 90 - m\angle E = 90 - 22.6 = 67.4$, and $\angle A \cong \angle D$.

$$\frac{AB}{DE} = \frac{13.0}{19.5} = \frac{2}{3} \qquad \frac{BC}{EF} = \frac{12.0}{18.0} = \frac{2}{3} \qquad \frac{CA}{FD} = \frac{5.00}{7.50} = \frac{2}{3}$$

Since the corresponding angles are congruent and the ratios of the lengths of corresponding sides are equal, the triangles are similar. ■

Exercises**Writing About Mathematics**

1. Are all squares similar? Justify your answer.
2. Are *any* two regular polygons similar? Justify your answer.

Developing Skills

3. What is the ratio of the lengths of corresponding sides of two congruent polygons?
4. Are all congruent polygons similar? Explain your answer.
5. Are all similar polygons congruent? Explain your answer.
6. What must be the constant of proportionality of two similar polygons in order for the polygons to be congruent?
7. The sides of a triangle measure 4, 9, and 11. If the shortest side of a similar triangle measures 12, find the measures of the remaining sides of this triangle.
8. The sides of a quadrilateral measure 12, 18, 20, and 16. The longest side of a similar quadrilateral measures 5. Find the measures of the remaining sides of this quadrilateral.

9. Triangle $ABC \sim \triangle A'B'C'$, and their ratio of similitude is $1 : 3$. If the measures of the sides of $\triangle ABC$ are represented by a , b , and c , represent the measures of the sides of the larger triangle, $\triangle A'B'C'$.

Applying Skills

10. Prove that any two equilateral triangles are similar.
11. Prove that any two regular polygons that have the same number of sides are similar.
12. In $\triangle ABC$, the midpoint of $\overline{AC}$ is M and the midpoint of $\overline{BC}$ is N .
- Show that $\triangle ABC \sim \triangle MNC$.
 - What is their ratio of similitude?
13. In $\triangle ABC$, the midpoint of $\overline{AC}$ is M , the midpoint of $\overline{BC}$ is P , the midpoint of $\overline{BC}$ is N , and the midpoint of $\overline{NC}$ is Q .
- Show that $\triangle ABC \sim \triangle PQC$.
 - What is their ratio of similitude?
14. Show that rectangle $ABCD$ is similar to rectangle $EFGH$ if $\frac{AB}{EF} = \frac{BC}{FG}$.
15. Show that parallelogram $KLMN$ is similar to parallelogram $PQRS$ if $m\angle K \cong m\angle P$ and $\frac{KL}{PQ} = \frac{LM}{QR}$.

12-4 PROVING TRIANGLES SIMILAR

We have proved triangles similar by proving that the corresponding angles are congruent and that the ratios of the lengths of corresponding sides are equal. It is possible to prove that when some of these conditions exist, all of these conditions necessary for triangles to be similar exist.

Hands-On Activity



For this activity, you may use a compass and ruler, or geometry software.

STEP 1. Draw any triangle, $\triangle ABC$.

STEP 2. Draw any line $\overline{DE}$ with $\frac{DE}{AB} = \frac{3}{1}$, that is, $DE = 3AB$.

STEP 3. Construct $\angle GDE \cong \angle A$ and $\angle HED \cong \angle B$. Let F be the intersection of $\overrightarrow{DG}$ and $\overrightarrow{EH}$.

- Find the measures of $\overline{AC}$, $\overline{BC}$, $\overline{DF}$, and $\overline{EF}$.
- Is $\frac{DF}{AC} = \frac{3}{1}$? Is $\frac{EF}{CB} = \frac{3}{1}$?
- Is $\triangle DEF \sim \triangle ABC$?
- Repeat this construction using a different ratio of similitude. Are the triangles similar?

Our observations from the activity on page 489 seem to suggest the following **postulate of similarity**.

Postulate 12.4

For any given triangle there exists a similar triangle with any given ratio of similitude.

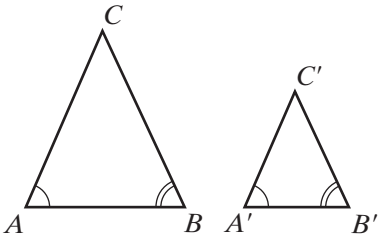
We can also prove the angle-angle or **AA triangle similarity** theorem.

Theorem 12.4

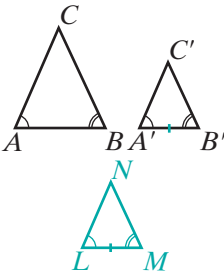
Two triangles are similar if two angles of one triangle are congruent to two corresponding angles of the other. (AA~)

Given $\triangle ABC$ and $\triangle A'B'C'$ with $\angle A \cong \angle A'$ and $\angle B \cong \angle B'$

Prove $\triangle A'B'C' \sim \triangle ABC$



Proof



Statement	Reason
1. Draw $\triangle LMN \sim \triangle ABC$ with $\frac{LM}{AB} = \frac{A'B'}{AB}$.	1. Postulate of similarity.
2. $\angle L \cong \angle A$ and $\angle M \cong \angle B$	2. If two triangles are similar, then their corresponding angles are congruent.
3. $\angle A \cong \angle A'$ and $\angle B \cong \angle B'$	3. Given.
4. $\angle L \cong \angle A'$ and $\angle M \cong \angle B'$	4. Substitution postulate.
5. $\frac{LM}{AB} = \frac{A'B'}{AB}$	5. Step 1.
6. $(A'B')(AB) = (AB)(LM)$	6. In any proportion, the product of the means is equal to the product of the extremes.
7. $A'B' = LM$	7. Division postulate.
8. $\triangle A'B'C' \cong \triangle LMN$	8. ASA (steps 4, 7).
9. $\triangle A'B'C' \sim \triangle LMN$	9. If two triangles are congruent, then they are similar.
10. $\triangle A'B'C' \sim \triangle ABC$	10. Transitive property of similarity (steps 1, 9).

We can also prove other theorems about similar triangles by construction, such as the side-side-side or **SSS similarity theorem**.

Theorem 12.5

Two triangles are similar if the three ratios of corresponding sides are equal. (SSS~)

Given $\triangle ABC$ and $\triangle A'B'C'$ with $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$.

Prove $\triangle A'B'C' \sim \triangle ABC$

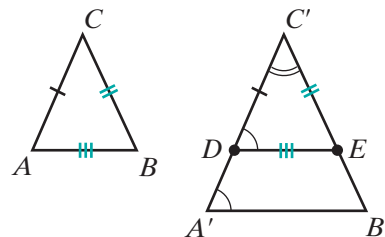
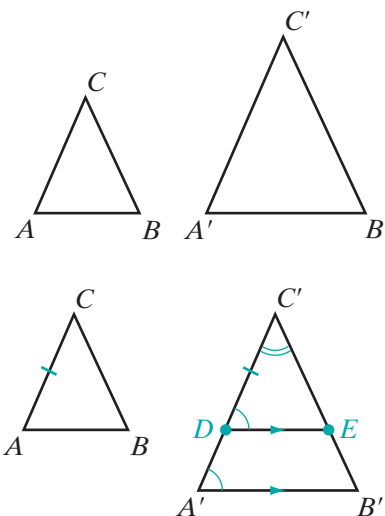
Proof We will construct a third triangle $\triangle DEC'$ that is similar to both $\triangle ABC$ and $\triangle A'B'C'$. By the transitive property of similarity, we can conclude that $\triangle A'B'C' \sim \triangle ABC$.

Let $AC < A'C'$. Choose point D on $\overline{A'C'}$ so that $DC' = AC$. Choose point E on $\overline{B'C'}$ so that $\overline{DE} \parallel \overline{A'B'}$. Corresponding angles of parallel lines are congruent, so $\angle C'DE \cong \angle A'$ and $\angle C' \cong \angle C'$. Therefore, $\triangle A'B'C' \sim \triangle DEC'$ by AA~. If two polygons are similar, then their corresponding sides are in proportion, so $\frac{C'D}{C'A'} = \frac{C'E}{C'B'}$.

Substituting $C'D = CA$ into the proportion gives $\frac{CA}{C'A'} = \frac{C'E}{C'B'}$.

We are given that $\frac{CA}{C'A'} = \frac{BC}{B'C'}$ so by the transitive property, $\frac{C'E}{C'B'} = \frac{BC}{B'C'}$. Therefore, $(C'E)(B'C') = (C'B')(BC)$ or $C'E = BC$.

By similar reasoning, we find that $DE = AB$. Therefore, $\triangle DEC' \cong \triangle ABC$ by SSS and $\triangle DEC' \sim \triangle ABC$. Then by the transitive property of similarity, $\triangle A'B'C' \sim \triangle ABC$. ■



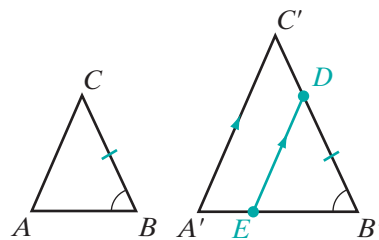
Theorem 12.6

Two triangles are similar if the ratios of two pairs of corresponding sides are equal and the corresponding angles included between these sides are congruent. (SAS~)

Given $\triangle ABC$ and $\triangle A'B'C'$ with $\frac{AB}{A'B'} = \frac{BC}{B'C'}$ and $\angle B \cong \angle B'$

Prove $\triangle ABC \sim \triangle A'B'C'$

Strategy The proof follows the same pattern as the previous theorem. Let $BC < B'C'$. Choose point D on $\overline{B'C'}$ so that $B'D = BC$. Choose point E on $\overline{A'B'}$ so that $\overline{DE} \parallel \overline{A'C'}$. First prove that $\triangle A'B'C' \sim \triangle EB'D$. Then use the given ratios to prove $EB' = AB$ and $\triangle EB'D \cong \triangle ABC$ by SAS.



We refer to Theorem 12.6 as the side-angle-side or **SAS similarity theorem**. The proof of this theorem will be left to the student. (See exercise 17.) As a consequence of these proofs, we have shown the following theorem to be true.

Theorem 12.7a

If a line is parallel to one side of a triangle and intersects the other two sides, then the points of intersection divide the sides proportionally.

The converse of this theorem is also true.

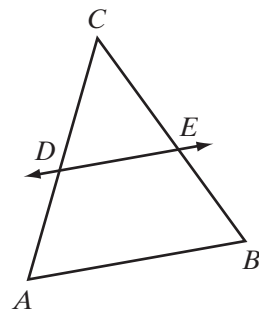
Theorem 12.7b

If the points at which a line intersects two sides of a triangle divide those sides proportionally, then the line is parallel to the third side.

Given $\triangle ABC$ with $\frac{CD}{CA} = \frac{CE}{CB}$

Prove $\overline{DE} \parallel \overline{AB}$

Proof Since $\frac{CD}{CA} = \frac{CE}{CB}$ and $\angle C \cong \angle C$, $\triangle ABC \sim \triangle DEC$ by SAS~. Corresponding angles of similar triangles are congruent, so $\angle CDE \cong \angle A$. As these are congruent corresponding angles, $\overline{DE} \parallel \overline{AB}$.



Theorems 12.7a and 12.7b can be written as a biconditional.

Theorem 12.7

A line is parallel to one side of a triangle and intersects the other two sides if and only if the points of intersection divide the sides proportionally.

EXAMPLE 1

The lengths of the sides of $\triangle PQR$ are $PQ = 15$ cm, $QR = 8$ cm, and $RP = 12$ cm. If $\triangle PQR \sim \triangle DEF$ and the length of the smallest side of $\triangle DEF$ is 6 centimeters, find the measures of the other two sides of $\triangle DEF$.

Solution Since the smallest side of $\triangle PQR$ is $\overline{QR}$ and $\overline{QR}$ corresponds to $\overline{EF}$, $EF = 6$ cm.

$$\frac{QR}{EF} = \frac{PQ}{DE}$$

$$\frac{8}{6} = \frac{15}{DE}$$

$$8DE = 90$$

$$DE = \frac{90}{8} = \frac{45}{4} = 11\frac{1}{4}$$

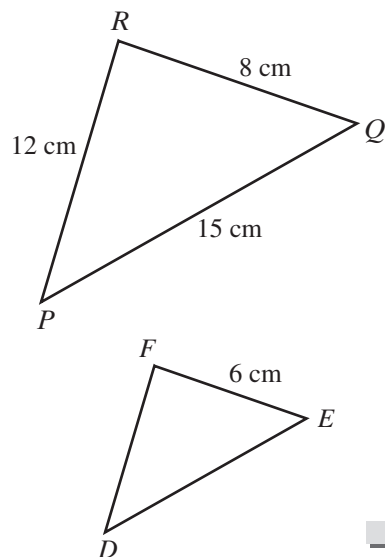
$$\frac{QR}{EF} = \frac{RP}{FD}$$

$$\frac{8}{6} = \frac{12}{FD}$$

$$8FD = 72$$

$$FD = \frac{72}{8} = 9$$

Answer $DE = 11\frac{1}{4}$ cm and $FD = 9$ cm



EXAMPLE 2

In $\triangle DEF$, a line is drawn parallel to $\overline{DE}$ that intersects $\overline{FD}$ at H and $\overline{FE}$ at G . If $FG = 8$, $GE = 12$, and $FD = 30$, find FH and HD .

Solution Since $\overline{GH}$ is parallel to $\overline{DE}$, H and G divide $\overline{FD}$ and $\overline{FE}$ proportionally, that is,

$$FH : HD = FG : GE.$$

Let $x = FH$.

Then $HD = FD - FH = 30 - x$.

$$\frac{FH}{HD} = \frac{FG}{GE}$$

$$\frac{x}{30 - x} = \frac{8}{12}$$

$$12x = 8(30 - x)$$

$$12x = 240 - 8x$$

$$20x = 240$$

$$x = 12$$

$$30 - x = 30 - 12$$

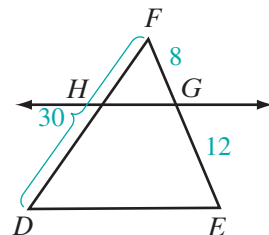
$$= 18$$

Check

$$\frac{FH}{HD} = \frac{FG}{GE}$$

$$\frac{12}{18} \stackrel{?}{=} \frac{8}{12}$$

$$\frac{2}{3} = \frac{2}{3} \quad \checkmark$$

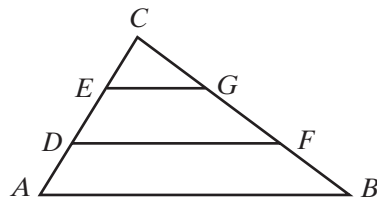


Answer $FH = 12$ and $HD = 18$

EXAMPLE 3

Given: $\overline{ADEC}$ and $\overline{BFGC}$ are two sides of $\triangle ABC$ with $AD = DE = EC$ and $BF = FG = GC$.

Prove: $\frac{AC}{DC} = \frac{BC}{FC}$ and $\triangle ABC \sim \triangle DFC$



Proof We are given $\overline{ADEC}$ and $AD = DE = EC$. Then $AC = AD + DE + EC$. By the substitution postulate, $AC = AD + AD + AD = 3AD$ and $DC = DE + EC = AD + AD = 2AD$.

We are also given $\overline{BFGC}$ and $BF = FG = GC$. Then $BC = BF + FG + GC$. By the substitution postulate, $BC = BF + BF + BF = 3BF$ and $FC = FG + GC = BF + BF = 2BF$.

Then, $\frac{AC}{DC} = \frac{3AD}{2AD} = \frac{3}{2}$ and $\frac{BC}{FC} = \frac{3BF}{2BF} = \frac{3}{2}$. Therefore, $\frac{AC}{DC} = \frac{BC}{FC}$.

In $\triangle ABC$ and $\triangle DFC$, $\frac{AC}{DC} = \frac{BC}{FC}$ and $\angle C \cong \angle C$. Therefore, $\triangle ABC \sim \triangle DFC$ by SAS~. ■

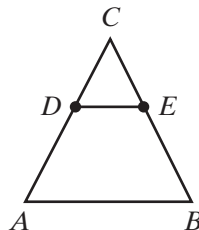
Exercises**Writing About Mathematics**

1. Javier said that if an acute angle of one right triangle is congruent to an acute angle of another right triangle, the triangles are similar. Do you agree with Javier? Explain why or why not.
2. Fatima said that since two triangles can be proven similar by AA~, it follows that two triangles can be proven similar by SS~. Explain why Fatima is incorrect.

Developing Skills

In 3–15, D is a point on $\overline{AC}$ and E is a point on $\overline{BC}$ of $\triangle ABC$ such that $\overline{DE} \parallel \overline{AB}$. (The figure is not drawn to scale.)

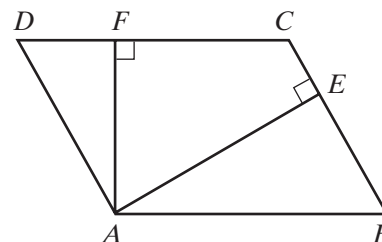
3. Prove that $\triangle ABC \sim \triangle DEC$.
4. If $CA = 8$, $AB = 10$, and $CD = 4$, find DE .
5. If $CA = 24$, $AB = 16$, and $CD = 9$, find DE .
6. If $CA = 16$, $AB = 12$, and $CD = 12$, find DE .
7. If $CE = 3$, $DE = 4$, and $CB = 9$, find AB .
8. If $CD = 8$, $DA = 2$, and $CB = 7.5$, find CE .
9. If $CD = 6$, $DA = 4$, and $DE = 9$, find AB .
10. If $CA = 35$, $DA = 10$, and $CE = 15$, find EB .



11. If $CA = 48$, $DA = 12$, and $CE = 30$, find EB .
12. If $CD = 15$, $DA = 9$, and $DE = 10$, find AB .
13. If $CE = 20$, $EB = 10$, and $AB = 45$, find DE .
14. If $CD = x$, $DE = x$, $DA = 5$, and $AB = 14$, find DE .
15. If $CD = 6$, $DE = x$, $DA = x - 1$, and $AB = 6$, DE and DA .

Applying Skills

16. Complete the proof of Theorem 12.5 (SSS~) by showing that $DE = AB$.
17. Prove Theorem 12.6, “Two triangles are similar if the ratios of two pairs of corresponding sides are equal and the corresponding angles included between these sides are congruent. (SAS~)”
18. Triangle ABC is an isosceles right triangle with $m\angle C = 90$ and $\overline{CD}$ bisects $\angle C$ and intersects $\overline{AB}$ at D . Prove that $\triangle ABC \sim \triangle ACD$.
19. Quadrilateral $ABCD$ is a trapezoid with $\overline{AB} \parallel \overline{CD}$. The diagonals $\overline{AC}$ and $\overline{BD}$ intersect at E . Prove that $\triangle ABE \sim \triangle CDE$.
20. Lines $\overleftrightarrow{AEB}$ and $\overleftrightarrow{CED}$ intersect at E and $\angle DAE \cong \angle BCE$. Prove that $\triangle ADE \sim \triangle CBE$.
21. In parallelogram $ABCD$ on the right, $\overline{AE} \perp \overline{BC}$ and $\overline{AF} \perp \overline{CD}$. Prove that $\triangle ABE \sim \triangle ADF$.
22. In the coordinate plane, the points $A(1, 2)$, $B(3, 2)$, $C(3, 6)$, $D(2, 6)$, and $E(2, 8)$ are the vertices of $\triangle ABC$ and $\triangle CDE$. Prove that $\triangle ABC \sim \triangle CDE$.
23. In the coordinate plane, the points $P(1, 1)$, $Q(3, 3)$, $R(3, 5)$, and $S(1, 5)$ are the vertices of $\triangle PQS$ and $\triangle QRS$ and $PQ = QS = 2\sqrt{2}$. Prove that $\triangle PQS \sim \triangle QRS$.
24. In the coordinate plane, the points $O(0, 0)$, $A(4, 0)$, and $B(0, 6)$ are the coordinates of $\triangle OAB$. The coordinates of C are $(4, 3)$, and D is the midpoint of $\overline{AB}$. Prove that $\triangle OAB \sim \triangle CDA$.
25. A pyramid with a triangular base is cut by a plane p parallel to the base. Prove that the triangle formed by the intersection of plane p with the lateral faces of the pyramid is similar to the base of the pyramid.



12-5 DILATIONS

In Chapter 6, we learned about dilations in the coordinate plane. In this section, we will continue to study dilations as they relate to similarity. Recall that a dilation is a transformation in the plane that preserves angle measure but not distance.

► A **dilation** of k is a transformation of the plane such that:

1. The image of point O , the center of dilation, is O .
2. When k is positive and the image of P is P' , then $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ are the same ray and $OP' = kOP$.
3. When k is negative and the image of P is P' , then $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ are opposite rays and $OP' = -kOP$.

When $|k| > 1$, the dilation is called an **enlargement**. When $0 < |k| < 1$, the dilation is called a **contraction**.

Recall also that in the coordinate plane, under a dilation of k with the center at the origin:

$$P(x, y) \rightarrow P'(kx, ky) \quad \text{or} \quad D_k(x, y) = (kx, ky)$$

For example, the image of $\triangle ABC$ is $\triangle A'B'C'$ under a dilation of $\frac{1}{2}$. The vertices of $\triangle ABC$ are $A(2, 6)$, $B(6, 4)$, and $C(4, 0)$. Under a dilation of $\frac{1}{2}$, the rule is

$$D_{\frac{1}{2}}(x, y) = \left(\frac{1}{2}x, \frac{1}{2}y\right)$$

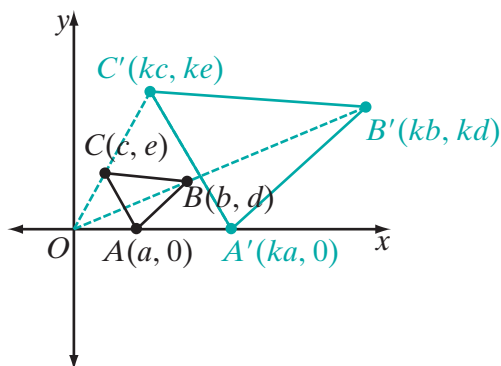
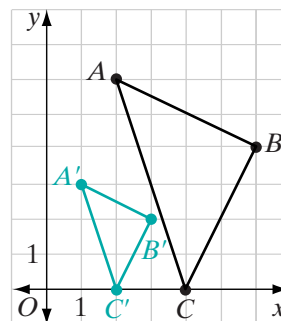
$$A(2, 6) \rightarrow A'(1, 3)$$

$$B(6, 4) \rightarrow B'(3, 2)$$

$$C(4, 0) \rightarrow C'(2, 0)$$

Notice that $\triangle ABC$ and $\triangle A'B'C'$ appear to be similar. We can use a general triangle to prove that for any dilation, the image of a triangle is a similar triangle.

Let $\triangle ABC$ be any triangle in the coordinate plane with $A(a, 0)$, $B(b, d)$, and $C(c, e)$. Under a dilation of k through the origin, the image of $\triangle ABC$ is $\triangle A'B'C'$, and the coordinates of $\triangle A'B'C'$ are $A'(ka, 0)$, $B'(kb, kd)$, and $C'(kc, ke)$.



$\begin{aligned} \text{Slope of } \overline{AB} &= \frac{d-0}{b-a} \\ &= \frac{d}{b-a} \\ \text{Slope of } \overline{A'B'} &= \frac{kd-0}{kb-ka} \\ &= \frac{k}{k} \left(\frac{d}{b-a} \right) \\ &= \frac{d}{b-a} \end{aligned}$ Therefore, $\overline{AB} \parallel \overline{A'B'}$.	$\begin{aligned} \text{Slope of } \overline{AC} &= \frac{e-0}{c-a} \\ &= \frac{e}{c-a} \\ \text{Slope of } \overline{A'C'} &= \frac{ke-0}{kc-ka} \\ &= \frac{k}{k} \left(\frac{e}{c-a} \right) \\ &= \frac{e}{c-a} \end{aligned}$ Therefore, $\overline{AC} \parallel \overline{A'C'}$.	$\begin{aligned} \text{Slope of } \overline{BC} &= \frac{d-e}{b-c} \\ \text{Slope of } \overline{B'C'} &= \frac{kd-ke}{kb-kc} \\ &= \frac{k}{k} \left(\frac{d-e}{b-c} \right) \\ &= \frac{d-e}{b-c} \end{aligned}$ Therefore, $\overline{BC} \parallel \overline{B'C'}$.
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We have shown that $\overline{AB} \parallel \overline{A'B'}$ and $\overline{AC} \parallel \overline{A'C'}$. Therefore, because they are corresponding angles of parallel lines:

$$m\angle OAB = m\angle OA'B'$$

$$m\angle OAC = m\angle OA'C'$$

$$m\angle OAB - m\angle OAC = m\angle OA'B' - m\angle OA'C'$$

$$m\angle BAC = m\angle B'A'C'$$

In a similar way we can prove that $\angle ACB \cong \angle A'C'B'$, and so $\triangle ABC \sim \triangle A'B'C'$ by AA~. Therefore, under a dilation, angle measure is preserved but distance is not preserved. Under a dilation of k , distance is changed by the factor k .

We have proved the following theorem:

Theorem 12.8

Under a dilation, angle measure is preserved.

We will now prove that under a dilation, midpoint and collinearity are preserved.

Theorem 12.9

Under a dilation, midpoint is preserved.

Proof: Under a dilation D_k :

$$A(a, c) \rightarrow A'(ka, kc)$$

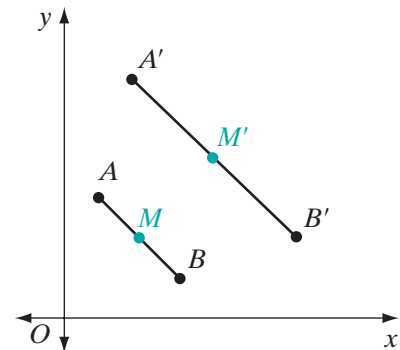
$$B(b, d) \rightarrow (kb, kd)$$

$$M\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \rightarrow M'\left(k\frac{a+b}{2}, k\frac{c+d}{2}\right)$$

The coordinates of the midpoint of $\overline{A'B'}$ are:

$$\left(\frac{ka+kb}{2}, \frac{kc+kd}{2}\right) \text{ or } \left(k\frac{a+b}{2}, k\frac{c+d}{2}\right)$$

Therefore, the image of M is the midpoint of the image of $\overline{AB}$, and midpoint is preserved. ■



Theorem 12.10

Under a dilation, collinearity is preserved.

Proof: Under a dilation D_k :

$$A(a, c) \rightarrow A'(ka, kc)$$

$$B(b, d) \rightarrow B'(kb, kd)$$

$$P(p, q) \rightarrow P'(kp, kq)$$

Since P is on $\overline{AB}$, the slope of $\overline{AP}$ is equal to the slope of $\overline{PB}$. Therefore:

$$\text{slope of } \overline{AP} = \text{slope of } \overline{PB}$$

$$\frac{c - q}{a - p} = \frac{q - d}{p - b}$$

P' will be on $\overline{A'B'}$ if and only if the slope of $\overline{A'P'}$ is equal to the slope of $\overline{P'B'}$.

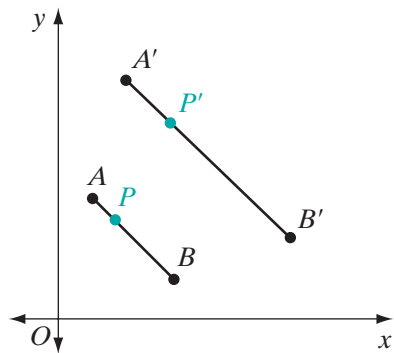
$$\text{slope of } \overline{A'P'} \stackrel{?}{=} \text{slope of } \overline{P'B'}$$

$$\frac{kc - kq}{ka - kp} \stackrel{?}{=} \frac{kq - kd}{kp - kb}$$

Since $\frac{c - q}{a - p} = \frac{d - q}{b - p}$ is true,

$$\frac{k}{k} \left(\frac{c - q}{a - p} \right) = \frac{k}{k} \left(\frac{q - d}{p - b} \right) \quad \text{or} \quad \frac{kc - kq}{ka - kp} = \frac{kd - kq}{kb - kp}$$

Thus, since we have shown that the slope of $\overline{A'P'}$ is equal to the slope of $\overline{P'B'}$, P' is on $\overline{A'B'}$ and collinearity is preserved. ■

**EXAMPLE I**

The coordinates of parallelogram $EFGH$ are $E(0, 0)$, $F(3, 0)$, $G(4, 2)$, and $H(1, 2)$. Under D_3 , the image of $EFGH$ is $E'F'G'H'$. Show that $E'F'G'H'$ is a parallelogram. Is parallelism preserved?

Solution $D_3(x, y) = (3x, 3y)$. Therefore, $E'(0, 0)$, $F'(9, 0)$, $G'(12, 6)$, and $H'(3, 6)$.

$$\text{slope of } \overline{E'F'} = \frac{0 - 0}{9 - 0} \quad \text{and} \quad \text{slope of } \overline{H'G'} = \frac{6 - 6}{12 - 3}$$

$$= 0 \quad \quad \quad = 0$$

$$\text{slope of } \overline{F'G'} = \frac{6 - 0}{12 - 9} \quad \text{and} \quad \text{slope of } \overline{E'H'} = \frac{6 - 0}{3 - 0}$$

$$= \frac{6}{3} \quad \quad \quad = \frac{6}{3}$$

$$= 2 \quad \quad \quad = 2$$

Since the slopes of the opposite sides of $E'F'G'H'$ are equal, the opposite sides are parallel and $E'F'G'H'$ is a parallelogram. Parallelism is preserved because the images of parallel lines are parallel. ■

EXAMPLE 2

Find the coordinates of Q' , the image of $Q(-3, 7)$ under the composition of transformations, $r_{y\text{-axis}} \circ D_{\frac{1}{2}}$.

Solution Perform the transformations from right to left.

The transformation at the right is to be performed first: $D_{\frac{1}{2}}(-3, 7) = \left(\frac{-3}{2}, \frac{7}{2}\right)$

Then perform the transformation on the left, using the result of the first transformation: $r_{y\text{-axis}}\left(\frac{-3}{2}, \frac{7}{2}\right) = \left(\frac{3}{2}, \frac{7}{2}\right)$

Answer $Q' = \left(\frac{3}{2}, \frac{7}{2}\right)$

**Exercises****Writing About Mathematics**

- Under D_k , $k > 0$, the image of $\triangle ABC$ is $\triangle A'B'C'$. Is $\frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{AC}{A'C'}$? Justify your answer.
- Under a dilation, the image of $A(3, 3)$ is $A'(4, 5)$ and the image of $B(4, 1)$ is $B'(6, 1)$. What are the coordinates of the center of dilation?

Developing Skills

In 3–6, use the rule $(x, y) \rightarrow \left(\frac{1}{3}x, \frac{1}{3}y\right)$ to find the coordinates of the image of each given point.

3. $(9, 6)$ 4. $(-5, 0)$ 5. $(18, 3)$ 6. $(-1, -7)$

In 7–10, find the coordinates of the image of each given point under D_3 .

7. $(8, 8)$ 8. $(2, 13)$ 9. $(-4, 7)$ 10. $\left(\frac{1}{3}, \frac{5}{8}\right)$

In 11–14, each given point is the image under D_{-2} . Find the coordinates of each preimage.

11. $(4, -2)$ 12. $(6, 8)$ 13. $(-3, -2)$ 14. $(20, 11)$

In 15–20, find the coordinates of the image of each given point under the given composition of transformations.

15. $D_3 \circ r_{x\text{-axis}}(2, 3)$ 16. $R_{180^\circ} \circ D_{-\frac{1}{2}}(4, 3)$ 17. $D_{\frac{5}{3}} \circ T_{5,3}(1, 1)$
 18. $r_{y\text{-axis}} \circ D_3(1, 2)$ 19. $T_{2,3} \circ D_{10\frac{1}{3}}(0, 0)$ 20. $D_{-2} \circ r_{y=x}(-3, -5)$

In 21–24, each transformation is the composition of a dilation and a reflection in either the x -axis or the y -axis. In each case, write a rule for composition of transformations for which the image of A is A' .

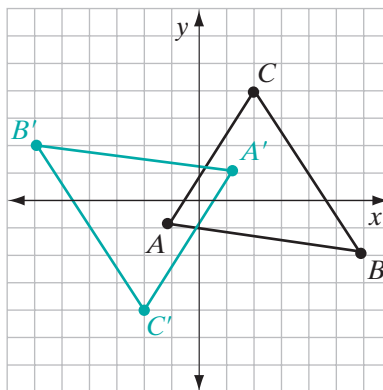
21. $A(3, 3) \rightarrow A'\left(\frac{9}{2}, -\frac{9}{2}\right)$

22. $A(5, -1) \rightarrow A'(20, 4)$

23. $A(20, 12) \rightarrow A'(-5, 3)$

24. $A(-50, 35) \rightarrow A'(10, 7)$

25. In the diagram, $\triangle A'B'C'$ is the image of $\triangle ABC$. Identify three specific transformations, or compositions of transformations, that can map $\triangle ABC$ to $\triangle A'B'C'$. Justify your answer.



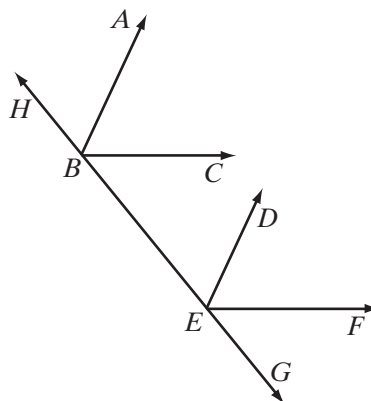
Applying Skills

26. If the coordinates of points A and B are $(0, 5)$ and $(5, 0)$, respectively, and A' and B' are the images of these points under D_{-3} , what type of quadrilateral is $ABA'B'$? Justify your answer.

27. Prove that if the sides of one angle are parallel to the sides of another angle, the angles are congruent.

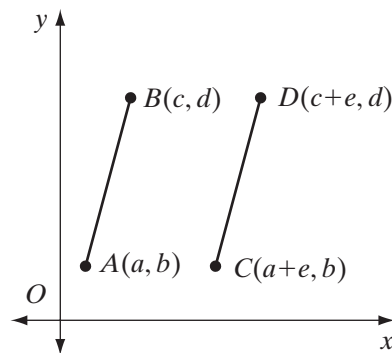
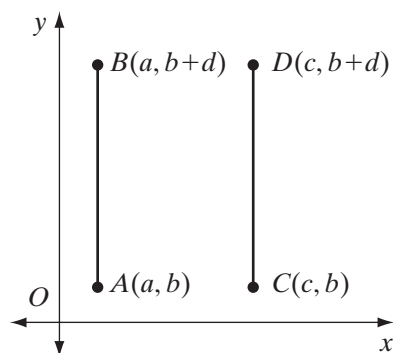
Given: $\overrightarrow{BA} \parallel \overrightarrow{ED}$, $\overrightarrow{BC} \parallel \overrightarrow{EF}$, and $\overrightarrow{BE} \parallel \overrightarrow{EG}$

Prove: $\angle ABC \cong \angle DEF$



28. The vertices of rectangle $ABCD$ are $A(2, -3)$, $B(4, -3)$, $C(4, 1)$, and $D(2, 1)$.
- Find the coordinates of the vertices of $A'B'C'D'$, the image of $ABCD$ under D_5 .
 - Show that $A'B'C'D'$ is a parallelogram.
 - Show that $ABCD \sim A'B'C'D'$.
 - Show that $\triangle ABC \sim \triangle A'B'C'$.

29. The vertices of octagon $ABCDEFGH$ are $A(2, 1)$, $B(1, 2)$, $C(-1, 2)$, $D(-2, 1)$, $E(-2, -1)$, $F(-1, -2)$, $G(1, -2)$, $H(2, -1)$.
- Draw $ABCDEFGH$ on graph paper.
 - Draw $A'B'C'D'E'F'G'H'$, the image of $ABCDEFGH$ under D_3 , on graph paper and write the coordinates of its vertices.
 - Find HA , BC , DE , FG .
 - Find $H'A'$, $B'C'$, $D'E'$, $F'G'$.
 - If $AB = CD = EF = GH = \sqrt{2}$, find $A'B'$, $C'D'$, $E'F'$, $G'H'$.
 - Are $ABCDEFGH$ and $A'B'C'D'E'F'G'H'$ similar polygons? Justify your answer.
30. Let the vertices of $\triangle ABC$ be $A(-2, 3)$, $B(-2, -1)$, and $C(3, -1)$.
- Find the area of $\triangle ABC$.
 - Find the area of the image of $\triangle ABC$ under D_3 .
 - Find the area of the image of $\triangle ABC$ under D_4 .
 - Find the area of the image of $\triangle ABC$ under D_5 .
 - Make a conjecture regarding how the area of a figure under a dilation D_k is related to the **constant of dilation k** .
31. Complete the following to prove that dilations preserve parallelism, that is, if $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, then the images of each line under a dilation D_k are also parallel.
- Let $\overline{AB}$ and $\overline{CD}$ be two vertical segments with endpoints $A(a, b)$, $B(a, b + d)$, $C(c, b)$, and $D(c, b + d)$. Under the dilation D_k , show that the images $\overline{A'B'}$ and $\overline{C'D'}$ are also parallel.
 - Let $\overline{AB}$ and $\overline{CD}$ be two nonvertical parallel segments with endpoints $A(a, b)$, $B(c, d)$, $C(a + e, b)$, and $D(c + e, d)$. Under the dilation D_k , show that the images $\overline{A'B'}$ and $\overline{C'D'}$ are also parallel.



I 2-6 PROPORTIONAL RELATIONS AMONG SEGMENTS RELATED TO TRIANGLES

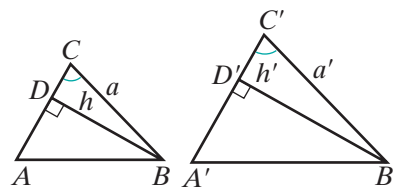
We have seen that, if two triangles are similar, their corresponding sides are in proportion. Other corresponding segments such as the altitudes, medians, and angle bisectors in similar triangles are also in proportion.

Theorem 12.11

If two triangles are similar, the lengths of corresponding altitudes have the same ratio as the lengths of any two corresponding sides.

Given $\triangle ABC \sim \triangle A'B'C'$ with the ratio of similitude $k : 1$, $\overline{BD} \perp \overline{AC}$, $\overline{B'D'} \perp \overline{A'C'}$, $BC = a$, $B'C' = a'$, $BD = h$, and $B'D' = h'$.

Prove $\frac{h}{h'} = \frac{a}{a'} = \frac{k}{1}$



Proof

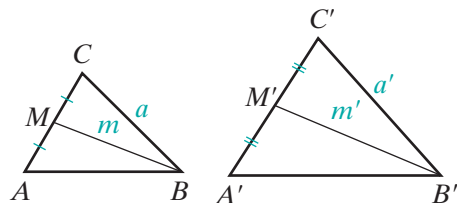
Statements	Reasons
1. $\triangle ABC \sim \triangle A'B'C'$	1. Given.
2. $\angle C \cong \angle C'$	2. If two triangles are similar, then their corresponding angles are congruent.
3. $\overline{BD} \perp \overline{AC}$ and $\overline{B'D'} \perp \overline{A'C'}$	3. Given.
4. $\angle BDC \cong \angle B'D'C'$	4. Perpendicular lines form right angles and all right angles are congruent.
5. $\triangle DBC \sim \triangle D'B'C'$	5. AA~.
6. $\frac{a}{a'} = \frac{k}{1}$	6. Given.
7. $\frac{h}{h'} = \frac{a}{a'}$	7. If two triangles are similar, then their corresponding sides are in proportion.
8. $\frac{h}{h'} = \frac{a}{a'} = \frac{k}{1}$	8. Transitive property. ■

We can prove related theorems for medians and angle bisectors of similar triangles.

Theorem 12.12

If two triangles are similar, the lengths of corresponding medians have the same ratio as the lengths of any two corresponding sides.

Given $\triangle ABC \sim \triangle A'B'C'$ with the ratio of similitude $k : 1$, M is the midpoint of $\overline{AC}$, M' is the midpoint of $\overline{A'C'}$, $BC = a$, $B'C' = a'$, $BM = m$, and $B'M' = m'$.



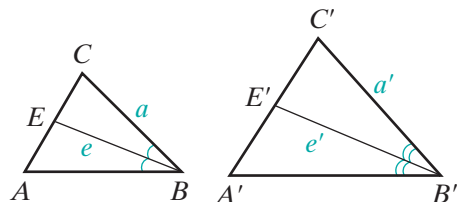
Prove $\frac{m}{m'} = \frac{a}{a'} = \frac{k}{1}$

Strategy Here we can use SAS~ to prove $\triangle BCM \sim \triangle B'C'M'$.

Theorem 12.13

If two triangles are similar, the lengths of corresponding angle bisectors have the same ratio as the lengths of any two corresponding sides.

Given $\triangle ABC \sim \triangle A'B'C'$ with the ratio of similitude $k : 1$, E is the point at which the bisector of $\angle B$ intersects $\overline{AC}$, E' is the point at which the bisector of $\angle B'$ intersects $\overline{A'C'}$, $BC = a$, $B'C' = a'$, $BE = e$, and $B'E' = e'$.



Prove $\frac{e}{e'} = \frac{a}{a'} = \frac{k}{1}$

Strategy Here we can use that halves of congruent angles are congruent and AA~ to prove $\triangle BCE \sim \triangle B'C'E'$.

The proofs of Theorems 12.12 and 12.13 are left to the student. (See exercises 10 and 11.)

EXAMPLE 1

Two triangles are similar. The sides of the smaller triangle have lengths of 4 meters, 6 meters, and 8 meters. The perimeter of the larger triangle is 63 meters. Find the length of the shortest side of the larger triangle.

- Solution** (1) In the smaller triangle, find the perimeter, p : $p = 4 + 6 + 8 = 18$
- (2) Let k be the constant of proportionality of the larger triangle to the smaller triangle. Let the measures of the sides of the larger triangle be a , b , and c . Set up proportions and solve for a , b , and c :

$$\begin{array}{lll} \frac{a}{4} = \frac{k}{1} & \frac{b}{6} = \frac{k}{1} & \frac{c}{8} = \frac{k}{1} \\ a = 4k & b = 6k & c = 8k \end{array}$$

- (3) Solve for k :

$$\begin{aligned} 4k + 6k + 8k &= 63 \\ 18k &= 63 \\ k &= 3.5 \end{aligned}$$

- (4) Solve for a , b , and c :

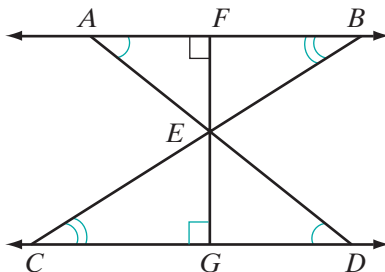
$$\begin{array}{lll} a = 4k & b = 6k & c = 8k \\ = 4(3.5) & = 6(3.5) & = 8(3.5) \\ = 14 & = 21 & = 28 \end{array}$$

Answer The length of the shortest side is 14 meters. ■

EXAMPLE 2

Given: $\overleftrightarrow{AFB} \parallel \overleftrightarrow{CGD}$, $\overline{AED}$ and $\overline{BEC}$ intersect at E , and $\overline{EF} \perp \overleftrightarrow{AFB}$.

Prove: $\triangle ABE \sim \triangle DCE$ and $\frac{AB}{DC} = \frac{EF}{EG}$.



Proof	Statements	Reasons
	1. $\overleftrightarrow{AFB} \parallel \overleftrightarrow{CGD}$	1. Given.
	2. $\angle EAB \cong \angle EDC$ and $\angle EBA \cong \angle ECD$	2. If two parallel lines are cut by a transversal, the alternate interior angles are congruent.
	3. $\triangle ABE \sim \triangle DCE$	3. AA $\sim$.
	4. $\overline{EF} \perp \overleftrightarrow{AFB}$	4. Given.
	5. $\overline{EG} \perp \overleftrightarrow{CGD}$	5. If a line is perpendicular to one of two parallel lines, it is perpendicular to the other.
	6. $\overline{EF}$ is an altitude from E in $\triangle ABE$. $\overline{EG}$ is an altitude from E in $\triangle DCE$.	6. Definition of an altitude of a triangle.
	7. $\frac{AB}{DC} = \frac{EF}{EG}$	7. If two triangles are similar, the lengths of corresponding altitudes have the same ratio as the lengths of any two corresponding sides.



Exercises

Writing About Mathematics

1. The lengths of the corresponding sides of two similar triangles are 10 and 25. Irena said that the ratio of similitude is 2 : 5. Jeff said that it is $\frac{2}{5}$: 1. Who is correct? Justify your answer.
2. Maya said that if the constant of proportionality of two similar triangles is k , then the ratio of the perimeters will be $3k$: 1 because $\frac{k}{1} + \frac{k}{1} + \frac{k}{1} = \frac{3k}{1}$. Do you agree with Maya? Explain why or why not.

Developing Skills

3. The ratio of similitude in two similar triangles is 5 : 1. If a side in the larger triangle measures 30 centimeters, find the measure of the corresponding side in the smaller triangle.
4. If the lengths of the sides of two similar triangles are in the ratio 4 : 3, what is the ratio of the lengths of a pair of corresponding altitudes, in the order given?
5. The lengths of two corresponding sides of two similar triangles are 18 inches and 12 inches. If an altitude of the smaller triangle has a length of 6 inches, find the length of the corresponding altitude of the larger triangle.

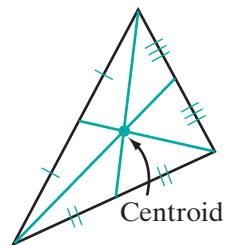
6. The constant of proportionality of two similar triangles is $\frac{4}{5}$. If the length of a median in the larger triangle is 15 inches, find the length of the corresponding median in the smaller triangle.
7. The ratio of the lengths of the corresponding sides of two similar triangles is 6 : 7. What is the ratio of the altitudes of the triangles?
8. Corresponding altitudes of two similar triangles have lengths of 9 millimeters and 6 millimeters. If the length of a median of the larger triangle is 24 millimeters, what is the length of a median of the smaller triangle?
9. In meters, the sides of a triangle measure 14, 18, and 12. The length of the longest side of a similar triangle is 21 meters.
 - a. Find the ratio of similitude of the two triangles.
 - b. Find the lengths of the other two sides of the larger triangle.
 - c. Find the perimeter of each triangle.
 - d. Is the ratio of the perimeters equal to the ratio of the lengths of the sides of the triangle?

Applying Skills

10. Prove Theorem 12.12, “If two triangles are similar, the lengths of corresponding medians have the same ratio as the lengths of any two corresponding sides.”
11. Prove Theorem 12.13, “If two triangles are similar, the lengths of corresponding angle bisectors have the same ratio as the lengths of any two corresponding sides.”
12. Prove that if two parallelograms are similar, then the ratio of the lengths of the corresponding diagonals is equal to the ratio of the lengths of the corresponding sides.
13. Prove that if two triangles are similar, then the ratio of their areas is equal to the square of their ratio of similitude.
14. The diagonals of a trapezoid intersect to form four triangles that have no interior points in common.
 - a. Prove that two of these four triangles are similar.
 - b. Prove that the ratio of similitude is the ratio of the length of the parallel sides.

12-7 CONCURRENCE OF THE MEDIANS OF A TRIANGLE

We proved in earlier chapters that the altitudes of a triangle are concurrent and that the angle bisectors of a triangle are concurrent. If we draw the three medians of a triangle, we see that they also seem to intersect in a point. This point is called the **centroid** of the triangle.

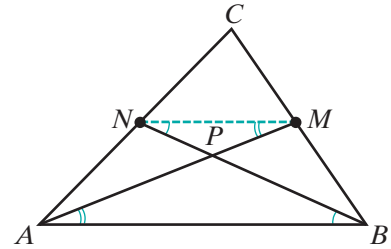


Theorem 12.14

Any two medians of a triangle intersect in a point that divides each median in the ratio 2 : 1.

Given $\overline{AM}$ and $\overline{BN}$ are medians of $\triangle ABC$ that intersect at P .

Prove $AP : MP = BP : NP = 2 : 1$



Proof

Statements	Reasons
1. $\overline{AM}$ and $\overline{BN}$ are the medians of $\triangle ABC$.	1. Given.
2. M is the midpoint of $\overline{BC}$ and N is the midpoint of $\overline{AC}$.	2. The median of a triangle is a line segment from a vertex to the midpoint of the opposite side.
3. Draw $\overline{MN}$.	3. Two points determine a line.
4. $\overline{MN} \parallel \overline{AB}$	4. The line joining the midpoints of two sides of a triangle is parallel to the third side.
5. $\angle MNB \cong \angle ABN$ and $\angle NMA \cong \angle BAM$	5. Alternate interior angles of parallel lines are congruent.
6. $\triangle MNP \sim \triangle ABP$	6. AA~.
7. $MN = \frac{1}{2}AB$	7. The length of the line joining the midpoints of two sides of a triangle is equal to one-half of the length of the third side.
8. $2MN = AB$	8. Multiplication postulate.
9. $AB : MN = 2 : 1$	9. If the products of two pairs of factors are equal, the factors of one pair can be the means and the factors of the other the extremes of a proportion.
10. $AP : MP = BP : NP = 2 : 1$	10. If two triangles are similar, the ratios of the lengths of the corresponding sides are equal.

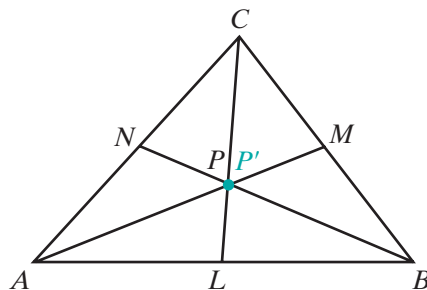
Theorem 12.15

The medians of a triangle are concurrent.

Given $\overline{AM}$, $\overline{BN}$, and $\overline{CL}$ are medians of $\triangle ABC$.

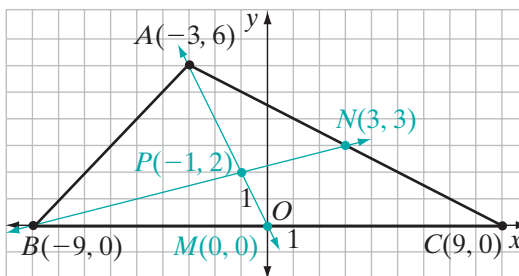
Prove $\overline{AM}$, $\overline{BN}$, and $\overline{CL}$ are concurrent.

Proof Let the intersection of $\overline{AM}$ and $\overline{BN}$ be P . Then P divides $\overline{AM}$ in the ratio $2 : 1$, that is, $AP : PM = 2 : 1$. Let $\overline{CL}$ intersect $\overline{AM}$ at P' . Then P' divides $\overline{AM}$ in the ratio $2 : 1$, that is, $AP' : P'M = 2 : 1$. Both P and P' are on the same line segment, $\overline{AM}$, and divide that line segment in the ratio $2 : 1$. Therefore, P and P' are the same point and the three medians of $\triangle ABC$ are concurrent. $\square$

**EXAMPLE I**

Find the coordinates of the centroid of the triangle whose vertices are $A(-3, 6)$, $B(-9, 0)$, and $C(9, 0)$.

Solution



- (1) Find the coordinates of the midpoint, M , of $\overline{BC}$ and of the midpoint, N , of $\overline{AC}$:

$$\begin{aligned}\text{coordinates of } M &= \left(\frac{-9 + 9}{2}, \frac{0 + 0}{2} \right) \\ &= (0, 0)\end{aligned}$$

$$\begin{aligned}\text{coordinates of } N &= \left(\frac{-3 + 9}{2}, \frac{6 + 0}{2} \right) \\ &= (3, 3)\end{aligned}$$

- (2) Find the equation of $\overleftrightarrow{AM}$ and the equation of $\overleftrightarrow{BN}$.

Equation of $\overleftrightarrow{AM}$:

$$\frac{y - 0}{x - 0} = \frac{6 - 0}{-3 - 0}$$

$$\frac{y}{x} = -2$$

$$y = -2x$$

Equation of $\overleftrightarrow{BN}$:

$$\frac{y - 0}{x - (-9)} = \frac{0 - 3}{-9 - 3}$$

$$\frac{y}{x + 9} = \frac{1}{4}$$

$$4y = x + 9$$

- (3) Find the coordinates of P , the point of intersection of $\overleftrightarrow{AM}$ and $\overleftrightarrow{BN}$:

Substitute $y = -2x$ into the equation $4y = x + 9$, and solve for x . Then find the corresponding value of y .

$$4(-2x) = x + 9$$

$$-8x = x + 9$$

$$-9x = 9$$

$$x = -1$$

$$y = -2x$$

$$y = -2(-1)$$

$$y = 2$$

Answer The coordinates of the centroid are $P(-1, 2)$. ■

We can verify the results of this example by showing that P is a point on the median from C :

- (1) The coordinates of L , the midpoint of $\overline{AB}$, are:

$$\left(\frac{-3 + (-9)}{2}, \frac{6 + 0}{2} \right) = (-6, 3)$$

- (2) The equation of $\overleftrightarrow{CL}$ is:

$$\frac{y - 0}{x - 9} = \frac{0 - 3}{9 - (-6)}$$

$$\frac{y}{x - 9} = \frac{-1}{5}$$

$$5y = -x + 9$$

- (3) $P(-1, 2)$ is a point on $\overleftrightarrow{CL}$:

$$5y = -x + 9$$

$$5(2) \stackrel{?}{=} -(-1) + 9$$

$$10 = 10 \quad \checkmark$$

Exercises

Writing About Mathematics

1. If $\overline{AM}$ and $\overline{BN}$ are two medians of $\triangle ABC$ that intersect at P , is P one of the points on $\overline{AM}$ that separate the segment into three congruent parts? Explain your answer.
2. Can the perpendicular bisector of a side of a triangle ever be the median to a side of a triangle? Explain your answer.

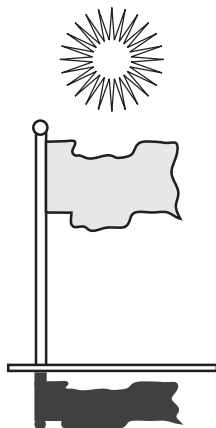
Developing Skills

In 3–10, find the coordinates of the centroid of each triangle with the given vertices.

- | | |
|------------------------------------|-------------------------------------|
| 3. $A(-3, 0), B(1, 0), C(-1, 6)$ | 4. $A(-5, -1), B(1, -1), C(1, 5)$ |
| 5. $A(-3, 3), B(3, -3), C(3, 9)$ | 6. $A(1, 2), B(7, 0), C(1, -2)$ |
| 7. $A(-1, 1), B(3, 1), C(1, 7)$ | 8. $A(-6, 2), B(0, 0), C(0, 10)$ |
| 9. $A(-2, -5), B(0, 1), C(-10, 1)$ | 10. $A(-1, -1), B(17, -1), C(5, 5)$ |

Applying Skills

11. The coordinates of a vertex of $\triangle ABC$ are $A(0, 6)$, and $AB = AC$.
 - a. If B and C are on the x -axis and $BC = 4$, find the coordinates of B and C .
 - b. Find the coordinates of the midpoint M of $\overline{AB}$ and of the midpoint N of $\overline{AC}$.
 - c. Find the equation of $\overleftrightarrow{CM}$.
 - d. Find the equation of $\overleftrightarrow{BN}$.
 - e. Find the coordinates of the centroid of $\triangle ABC$.
12. The coordinates of the midpoint of $\overline{AB}$ of $\triangle ABC$ are $M(3, 0)$ and the coordinates of the centroid are $P(0, 0)$. If $\triangle ABC$ is isosceles and $AB = 6$, find the coordinates of A , B , and C .

12-8 PROPORTIONS IN A RIGHT TRIANGLE**Projection of a Point or of a Line Segment on a Line**

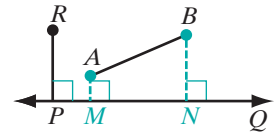
Whenever the sun is shining, any object casts a shadow. If the sun were directly overhead, the projection of an object would be suggested by the shadow of that object.

DEFINITION

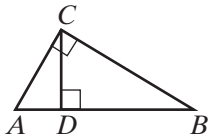
The **projection of a point on a line** is the foot of the perpendicular drawn from that point to the line.

The **projection of a segment on a line**, when the segment is not perpendicular to the line, is the segment whose endpoints are the projections of the endpoints of the given line segment on the line.

In the figure, $\overline{MN}$ is the projection of $\overline{AB}$ on $\overleftrightarrow{PQ}$. The projection of R on $\overleftrightarrow{PQ}$ is P . If $\overline{PR} \perp \overleftrightarrow{PQ}$, the projection of $\overline{PR}$ on $\overleftrightarrow{PQ}$ is P .



Proportions in the Right Triangle



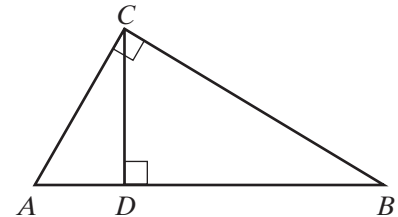
In the figure, $\triangle ABC$ is a right triangle, with the right angle at C . Altitude $\overline{CD}$ is drawn to hypotenuse $\overline{AB}$ so that two smaller triangles are formed, $\triangle ACD$ and $\triangle CBD$. Since $\overline{CD} \perp \overline{AB}$, $\angle CDA$ and $\angle CDB$ are right angles. The projection of $\overline{AC}$ on $\overline{AB}$ is $\overline{AD}$ and the projection of $\overline{BC}$ on $\overline{AB}$ is $\overline{BD}$. We want to prove that the three right triangles, $\triangle ABC$, $\triangle ACD$, and $\triangle CBD$, are similar triangles and, because they are similar triangles, the lengths of corresponding sides are in proportion.

Theorem 12.16

The altitude to the hypotenuse of a right triangle divides the triangle into two triangles that are similar to each other and to the original triangle.

Given $\triangle ABC$ with $\angle ACB$ a right angle and altitude $\overline{CD} \perp \overline{AB}$ at D .

Prove $\triangle ABC \sim \triangle ACD \sim \triangle CBD$



Proof	Statements	Reasons
	1. $\angle ACB$ is a right angle.	1. Given.
	2. $\overline{CD} \perp \overline{AB}$	2. Given.
	3. $\angle ADC$ and $\angle BDC$ are right angles.	3. Perpendicular lines intersect to form right angles.
	4. $\angle ACB \cong \angle ADC \cong \angle BDC$	4. All right angles are congruent.
	5. $\angle A \cong \angle A$ and $\angle B \cong \angle B$	5. Reflexive property of congruence.
	6. $\triangle ABC \sim \triangle ACD$ and $\triangle ABC \sim \triangle CBD$	6. AA~.
	7. $\triangle ABC \sim \triangle ACD \sim \triangle CBD$	7. Transitive property of similarity.



Now that we have proved that these triangles are similar, we can prove that the lengths of corresponding sides are in proportion. Recall that if the means of a proportion are equal, either mean is called the *mean proportional* between the extremes.

Corollary 12.16a

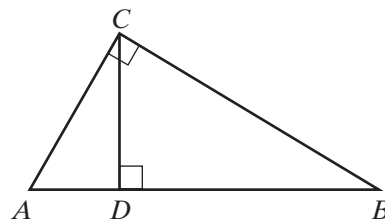
The length of each leg of a right triangle is the mean proportional between the length of the projection of that leg on the hypotenuse and the length of the hypotenuse.

Given $\triangle ABC$ with $\angle ACB$ a right angle and altitude $\overline{CD} \perp \overline{AB}$ at D

Prove $\frac{AB}{AC} = \frac{AC}{AD}$ and $\frac{AB}{BC} = \frac{BC}{BD}$

Proof The lengths of the corresponding sides of similar triangles are in proportion.

Therefore, since $\triangle ABC \sim \triangle ACD$,
 $\frac{AB}{AC} = \frac{AC}{AD}$ and since $\triangle ABC \sim \triangle CBD$,
 $\frac{AB}{BC} = \frac{BC}{BD}$.

**Corollary 12.16b**

The length of the altitude to the hypotenuse of a right triangle is the mean proportional between the lengths of the projections of the legs on the hypotenuse.

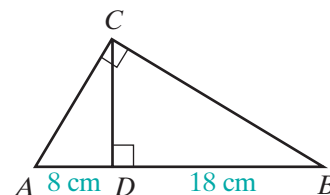
Proof: The lengths of the corresponding sides of similar triangles are in proportion. Therefore, since $\triangle ACD \sim \triangle CBD$, $\frac{AD}{CD} = \frac{CD}{BD}$.

EXAMPLE 1

In right triangle ABC , altitude $\overline{CD}$ is drawn to hypotenuse $\overline{AB}$. If $AD = 8$ cm and $DB = 18$ cm, find: **a.** AC **b.** BC **c.** CD

Solution

$$\begin{aligned} AB &= AD + DB \\ &= 8 + 18 \\ &= 26 \end{aligned}$$



Since $\overline{CD}$ is the altitude to the hypotenuse of right $\triangle ABC$, then:

$$\begin{array}{lll}
 \frac{AB}{AC} = \frac{AC}{AD} & \frac{AB}{BC} = \frac{BC}{BD} & \frac{AD}{CD} = \frac{CD}{DB} \\
 \frac{26}{AC} = \frac{AC}{8} & \frac{26}{BC} = \frac{BC}{18} & \frac{8}{CD} = \frac{CD}{18} \\
 (AC)^2 = 208 & (BC)^2 = 468 & (CD)^2 = 144 \\
 AC = \sqrt{208} & BC = \sqrt{468} & CD = \sqrt{144} \\
 = \sqrt{16}\sqrt{13} & = \sqrt{36}\sqrt{13} & = 12 \\
 = 4\sqrt{13} & = 6\sqrt{13} &
 \end{array}$$

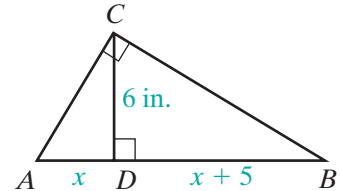
Answers a. $4\sqrt{13}$ cm b. $6\sqrt{13}$ cm c. 12 cm

EXAMPLE 2

The altitude to the hypotenuse of right triangle ABC separates the hypotenuse into two segments. The length of one segment is 5 inches more than the measure of the other. If the length of the altitude is 6 inches, find the length of the hypotenuse.

Solution Let x = the measure of the shorter segment.

Then $x + 5$ = the measure of the longer segment.



- (1) The length of the altitude is the mean proportional between the lengths of the segments of the hypotenuse:

$$\frac{x}{6} = \frac{6}{x + 5}$$

- (2) Set the product of the means equal to the product of the extremes:
- (3) Write the equation in standard form:
- (4) Factor the left side:
- (5) Set each factor equal to 0 and solve for x . Reject the negative root:
- (6) The length of the hypotenuse is the sum of the lengths of the segments:

$$x(x + 5) = 36$$

$$x^2 + 5x = 36$$

$$x^2 + 5x - 36 = 0$$

$$(x - 4)(x + 9) = 0$$

$$x - 4 = 0 \quad | \quad x + 9 = 0$$

$$x = 4 \quad | \quad x = -9 \text{ reject}$$

$$x + x + 5 = 4 + 4 + 5$$

$$= 13 \text{ in.}$$

Answer The length of the hypotenuse is 13 inches.

Exercises

Writing About Mathematics

1. When altitude $\overline{CD}$ is drawn to the hypotenuse of right triangle ABC , it is possible that $\triangle ACD$ and $\triangle BCD$ are congruent as well as similar. Explain when $\triangle ACD \cong \triangle BCD$.
2. The altitude to the hypotenuse of right $\triangle RST$ separates the hypotenuse, $\overline{RS}$, into two congruent segments. What must be true about $\triangle RST$?

Developing Skills

In 3–12, $\triangle ABC$ is a right triangle with $\angle ACB$ the right angle. Altitude $\overline{CD}$ intersects $\overline{AB}$ at D . In each case find the required length.

- | | |
|---|---|
| 3. If $AD = 3$ and $CD = 6$, find DB . | 4. If $AB = 8$ and $AC = 4$, find AD . |
| 5. If $AC = 10$ and $AD = 5$, find AB . | 6. If $AC = 6$ and $AB = 9$, find AD . |
| 7. If $AD = 4$ and $DB = 9$, find CD . | 8. If $DB = 4$ and $BC = 10$, find AB . |
| 9. If $AD = 3$ and $DB = 27$, find CD . | 10. If $AD = 2$ and $AB = 18$, find AC . |
| 11. If $DB = 8$ and $AB = 18$, find BC . | 12. If $AD = 3$ and $DB = 9$, find AC . |

Applying Skills

In 13–21, the altitude to the hypotenuse of a right triangle divides the hypotenuse into two segments.

13. If the lengths of the segments are 5 inches and 20 inches, find the length of the altitude.
14. If the length of the altitude is 8 feet and the length of the shorter segment is 2 feet, find the length of the longer segment.
15. If the ratio of the lengths of the segments is 1:9 and the length of the altitude is 6 meters, find the lengths of the two segments.
16. The altitude drawn to the hypotenuse of a right triangle divides the hypotenuse into two segments of lengths 4 and 5. What is the length of the altitude?
17. If the length of the altitude to the hypotenuse of a right triangle is 8, and the length of the hypotenuse is 20, what are the lengths of the segments of the hypotenuse? (Let x and $20 - x$ be the lengths of the segments of the hypotenuse.)
18. The altitude drawn to the hypotenuse of a right triangle divides the hypotenuse into segments of lengths 2 and 16. What are the lengths of the legs of the triangle?
19. In a right triangle whose hypotenuse measures 50 centimeters, the shorter leg measures 30 centimeters. Find the measure of the projection of the shorter leg on the hypotenuse.
20. The segments formed by the altitude to the hypotenuse of right triangle ABC measure 8 inches and 10 inches. Find the length of the shorter leg of $\triangle ABC$.
21. The measures of the segments formed by the altitude to the hypotenuse of a right triangle are in the ratio 1 : 4. The length of the altitude is 14.
 - a. Find the measure of each segment.
 - b. Express, in simplest radical form, the length of each leg.

12-9 PYTHAGOREAN THEOREM

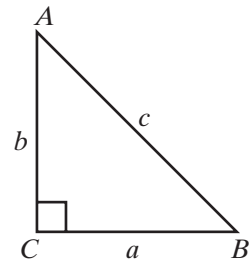
The theorems that we proved in the last section give us a relationship between the length of a leg of a right triangle and the length of the hypotenuse. These proportions are the basis for a proof of the **Pythagorean Theorem**, which was studied in earlier courses.

Theorem 12.17a

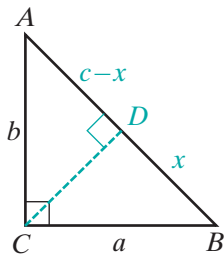
If a triangle is a right triangle, then the square of the length of the longest side is equal to the sum of the squares of the lengths of the other two sides (the legs).

Given $\triangle ABC$ is a right triangle with $\angle ACB$ the right angle, c is the length of the hypotenuse, a and b are the lengths of the legs.

Prove $c^2 = a^2 + b^2$



Proof



Statements

Reasons

1. $\triangle ABC$ is a right triangle with $\angle ACB$ the right angle.

1. Given.

2. Draw $\overline{CD} \perp \overline{AB}$.
Let $BD = x$ and $AD = c - x$.

2. From a point not on a given line, one and only one perpendicular can be drawn to the given line.

3. $\frac{c}{a} = \frac{a}{x}$ and $\frac{c}{b} = \frac{b}{c-x}$

3. The length of each leg of a right triangle is the mean proportional between the length of the projection of that leg on the hypotenuse and the length of the hypotenuse.

4. $cx = a^2$ and $c(c-x) = b^2$
 $c^2 - cx = b^2$

4. In a proportion, the product of the means is equal to the product of the extremes.

5. $cx + c^2 - cx = a^2 + b^2$
 $c^2 = a^2 + b^2$

5. Addition postulate.



The Converse of the Pythagorean Theorem

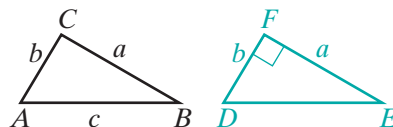
If we know the lengths of the three sides of a triangle, we can determine whether the triangle is a right triangle by using the converse of the Pythagorean Theorem.

Theorem 12.17b

If the square of the length of one side of a triangle is equal to the sum of the squares of the lengths of the other two sides, then the triangle is a right triangle.

Given $\triangle ABC$ with $AB = c$, $BC = a$, $CA = b$, and $c^2 = a^2 + b^2$

Prove $\triangle ABC$ is a right triangle with $\angle C$ the right angle.



Proof Draw $\triangle DEF$ with $EF = a$, $FD = b$, and $\angle F$ a right angle. Then $DE^2 = a^2 + b^2$, $DE^2 = c^2$ and $DE = c$. Therefore, $\triangle ABC \cong \triangle DEF$ by SSS. Corresponding angles of congruent triangles are congruent, so $\angle C \cong \angle F$ and $\angle C$ is a right angle. Triangle ABC is a right triangle. ■

We can state Theorems 12.17a and 12.17b as a single theorem.

Theorem 12.17

A triangle is a right triangle if and only if the square of the length of the longest side is equal to the sum of the squares of the lengths of the other two sides.

EXAMPLE 1

What is the length of the altitude to the base of an isosceles triangle if the length of the base is 18 centimeters and the length of a leg is 21 centimeters? Round your answer to the nearest centimeter.

Solution The altitude to the base of an isosceles triangle is perpendicular to the base and bisects the base. In $\triangle ABC$, $\overline{CD}$ is the altitude to the base $\overline{AB}$, $\overline{AC}$ is the hypotenuse of right triangle ACD , $AD = 9.0$ cm, and $AC = 21$ cm.

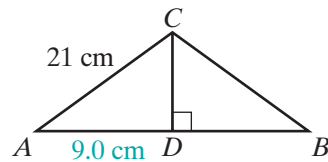
$$AD^2 + CD^2 = AC^2$$

$$9^2 + CD^2 = 21^2$$

$$81 + CD^2 = 441$$

$$CD^2 = 360$$

$$CD = \sqrt{360} = \sqrt{36}\sqrt{10} = 6\sqrt{10} \approx 19$$



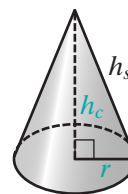
Answer The length of the altitude is approximately 19 centimeters. ■

EXAMPLE 2

When a right circular cone is cut by a plane through the vertex and perpendicular to the base of the cone, the cut surface is an isosceles triangle. The length of the hypotenuse of the triangle is the slant height of the cone, the length one of the legs is the height of the cone, and the length of the other leg is the radius of the base of the cone. If a cone has a height of 24 centimeters and the radius of the base is 10 centimeters, what is the slant height of the cone?

Solution Use the Pythagorean Theorem:

$$\begin{aligned}(h_s)^2 &= (h_c)^2 + r^2 \\(h_s)^2 &= 24^2 + 10^2 \\(h_s)^2 &= 676 \\h_s &= 26 \text{ cm} \quad \text{Answer}\end{aligned}$$



Pythagorean Triples

When three integers can be the lengths of the sides of a right triangle, this set of numbers is called a **Pythagorean triple**. The most common Pythagorean triple is 3, 4, 5:

$$3^2 + 4^2 = 5^2$$

If we multiply each number of a Pythagorean triple by some positive integer x , then the new triple created is also a Pythagorean triple because it will satisfy the relation $a^2 + b^2 = c^2$. For example:

If $\{3, 4, 5\}$ is a Pythagorean triple, then $\{3x, 4x, 5x\}$ is also a Pythagorean triple for a similar triangle where the ratio of similitude of the second triangle to the first triangle is $x : 1$.

Let $x = 2$. Then $\{3x, 4x, 5x\} = \{6, 8, 10\}$ and $6^2 + 8^2 = 10^2$.

Let $x = 3$. Then $\{3x, 4x, 5x\} = \{9, 12, 15\}$, and $9^2 + 12^2 = 15^2$.

Let $x = 10$. Then $\{3x, 4x, 5x\} = \{30, 40, 50\}$, and $30^2 + 40^2 = 50^2$.

Here are other examples of Pythagorean triples that occur frequently:

$\{5, 12, 13\}$ or, in general, $\{5x, 12x, 13x\}$ where x is a positive integer.

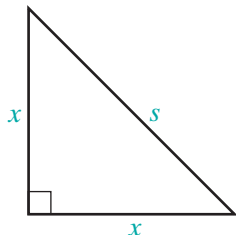
$\{8, 15, 17\}$ or, in general, $\{8x, 15x, 17x\}$ where x is a positive integer.

The 45-45-Degree Right Triangle

The legs of an isosceles right triangle are congruent. The measure of each acute angles of an isosceles right triangle is 45° . If two triangles are isosceles right triangles then they are similar by $AA \sim$. An isosceles right triangle is called a **45-45-degree right triangle**.

When a diagonal of a square is drawn, the square is separated into two isosceles right triangles. We can express the length of a leg of the isosceles right triangle in terms of the length of the hypotenuse or the length of the hypotenuse in terms of the length of a leg.

Let s be the length of the hypotenuse of an isosceles right triangle and x be the length of each leg. Use the Pythagorean Theorem to set up two equalities. Solve one for x and the other for s :



Solve for x :

$$a^2 + b^2 = c^2$$

$$x^2 + x^2 = s^2$$

$$2x^2 = s^2$$

$$x^2 = \frac{s^2}{2}$$

$$x = \sqrt{\frac{s^2}{2}}$$

$$x = \sqrt{\frac{s^2}{2} \cdot \frac{2}{2}}$$

$$x = \frac{\sqrt{2}}{2}s$$

Solve for s :

$$c^2 = a^2 + b^2$$

$$s^2 = x^2 + x^2$$

$$s^2 = 2x^2$$

$$s = x\sqrt{2}$$

The 30-60-Degree Right Triangle

An altitude drawn to any side of an equilateral triangle bisects the base and separates the triangle into two congruent right triangles. Since the measure of each angle of an equilateral triangle is 60° , the measure of one of the acute angles of a right triangle formed by the altitude is 60° and the measure of the other acute angle is 30° . Each of the congruent right triangles formed by drawing an altitude to a side of an equilateral triangle is called a **30-60-degree right triangle**. If two triangles are 30-60-degree right triangles, then they are similar by $AA \sim$.

In the diagram, $\triangle ABC$ is an equilateral triangle with s the length of each side and h the length of an altitude. Then s is the length of the hypotenuse of the 30-60-degree triangle and h is the length of a leg. In the diagram, $\overline{CD} \perp \overline{AB}$, $AC = s$, $AD = \frac{s}{2}$, and $CD = h$.

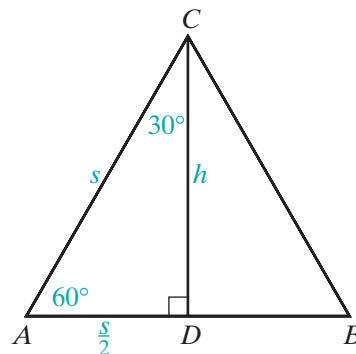
$$a^2 + b^2 = c^2$$

$$\left(\frac{s}{2}\right)^2 + h^2 = s^2$$

$$\frac{s^2}{4} + h^2 = \frac{4}{4}s^2$$

$$h^2 = \frac{3}{4}s^2$$

$$h = \frac{\sqrt{3}}{2}s$$

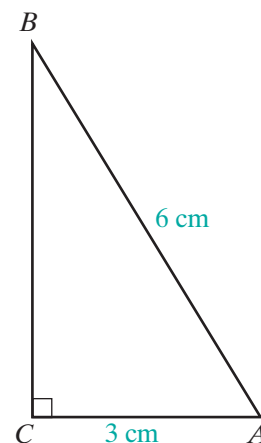


EXAMPLE 3

In right triangle ABC , the length of the hypotenuse, $\overline{AB}$, is 6 centimeters and the length of one leg is 3 centimeters. Find the length of the other leg.

Solution

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + (3)^2 &= (6)^2 \\ a^2 + 9 &= 36 \\ a^2 &= 27 \\ a &= \sqrt{27} = \sqrt{9 \cdot 3} = 3\sqrt{3} \quad \text{Answer} \end{aligned}$$



Note: The measure of one leg is one-half the measure of the hypotenuse and the length of the other leg is $\frac{\sqrt{3}}{2}$ times the length of the hypotenuse. Therefore, this triangle is a 30-60-degree right triangle. We can use a calculator to verify this.



Recall that $\tan A = \frac{\text{opp}}{\text{adj}} = \frac{BC}{AC} = \frac{3\sqrt{3}}{3} = \sqrt{3}$. Use a graphing calculator to find the measure of $\angle A$.

ENTER: **2nd** **TAN⁻¹** **2nd** **$\sqrt{}$** **3** **ENTER**

The calculator will return 60 as $m\angle A$. Therefore, $m\angle B = 30$.

Exercises

Writing About Mathematics

1. Ira said that if the lengths of the sides of an obtuse triangle $\triangle ABC$ are a , b , and c with c opposite the obtuse angle, then $c^2 > a^2 + b^2$. Do you agree with Ira? Explain why or why not. (*Hint:* Make use of the altitude from one of the acute angles.)
2. Sean said that if the measures of the diagonals of a parallelogram are 6 and 8 and the measure of one side of the parallelogram is 5 then the parallelogram is a rhombus. Do you agree with Sean? Explain why or why not.

Developing Skills

In 3–8, in each case the lengths of three sides of a triangle are given. Tell whether each triangle is a right triangle.

3. 6, 8, 10 4. 7, 8, 12 5. 5, 7, 8 6. 15, 36, 39 7. 14, 48, 50 8. $2, 2\sqrt{3}, 4$

9. Find, to the nearest tenth of a centimeter, the length of a diagonal of a square if the measure of one side is 8.0 centimeters.
10. Find the length of the side of a rhombus whose diagonals measure 40 centimeters and 96 centimeters.
11. The length of a side of a rhombus is 10 centimeters and the length of one diagonal is 120 millimeters. Find the length of the other diagonal.
12. The length of each side of a rhombus is 13 feet. If the length of the shorter diagonal is 10 feet, find the length of the longer diagonal.
13. Find the length of the diagonal of a rectangle whose sides measure 24 feet by 20 feet.
14. The diagonal of a square measures 12 feet.
 - a. What is the exact measure of a side of the square?
 - b. What is the area of the square?
15. What is the slant height of a cone whose height is 36 centimeters and whose radius is 15 centimeters?
16. One side of a rectangle is 9 feet longer than an adjacent side. The length of the diagonal is 45 feet. Find the dimensions of the rectangle.
17. One leg of a right triangle is 1 foot longer than the other leg. The hypotenuse is 9 feet longer than the shorter leg. Find the length of the sides of the triangle.

Applying Skills

18. Marvin wants to determine the edges of a rectangular garden that is to be 10 feet by 24 feet. He has no way of determining the measure of an angle but he can determine lengths very accurately. He takes a piece of cord that is 60 feet long and makes a mark at 10 feet and at 34 feet from one end. Explain how the cord can help him to make sure that his garden is a rectangle.
19. A plot of land is in the shape of an isosceles trapezoid. The lengths of the parallel sides are 109 feet and 95 feet. The length of each of the other two sides is 25 feet. What is the area of the plot of land?
20. From a piece of cardboard, Shanti cut a semicircle with a radius of 10 inches. Then she used tape to join one half of the diameter along which the cardboard had been cut to the other half, forming a cone. What is the height of the cone that Shanti made?
21. The lengths of two adjacent sides of a parallelogram are 21 feet and 28 feet. If the length of a diagonal of the parallelogram is 35 feet, show that the parallelogram is a rectangle.
22. The lengths of the diagonals of a parallelogram are 140 centimeters and 48 centimeters. The length of one side of the parallelogram is 74 centimeters. Show that the parallelogram is a rhombus.

- 23.** A young tree is braced by wires that are 9 feet long and fastened at a point on the trunk of the tree 5 feet from the ground. Find to the nearest tenth of a foot how far from the foot of the tree the wires should be fastened to the ground in order to be sure that the tree will be perpendicular to the ground.
- 24.** The length of one side of an equilateral triangle is 12 feet. What is the distance from the centroid of the triangle to a side? (Express the answer in simplest radical form.)

12-10 THE DISTANCE FORMULA

When two points in the coordinate plane are on the same vertical line, they have the same x -coordinate and the distance between them is the absolute value of the difference of their y -coordinates. In the diagram, the coordinates of A are $(4, 8)$ and the coordinates of C are $(4, 2)$.

$$\begin{aligned} CA &= |8 - 2| \\ &= 6 \end{aligned}$$

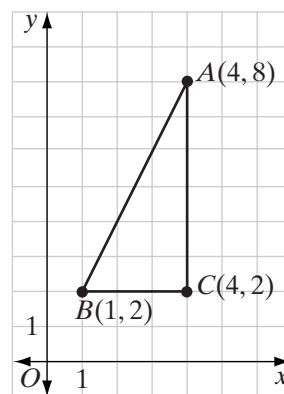
When two points in the coordinate plane are on the same horizontal line, they have the same y -coordinate and the distance between them is the absolute value of the difference of their x -coordinates. In the diagram, the coordinates of B are $(1, 2)$ and the coordinates of C are $(4, 2)$.

$$\begin{aligned} CB &= |1 - 4| \\ &= 3 \end{aligned}$$

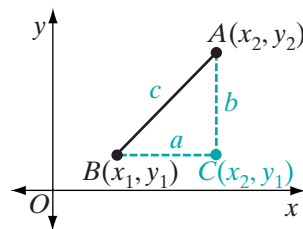
In $\triangle ABC$, $\angle C$ is a right angle and $\overline{AB}$ is the hypotenuse of a right triangle. Using the Pythagorean Theorem, we can find AB :

$$\begin{aligned} AB^2 &= CA^2 + CB^2 \\ AB^2 &= 6^2 + 3^2 \\ AB^2 &= 36 + 9 \\ AB &= \sqrt{45} \\ AB &= 3\sqrt{5} \end{aligned}$$

This example suggests a method that can be used to find a formula for the length of any segment in the coordinate plane.



Let $B(x_1, y_1)$ and $A(x_2, y_2)$ be any two points in the coordinate plane. From A draw a vertical line and from B draw a horizontal line. Let the intersection of these two lines be C . The coordinates of C are (x_2, y_1) . Let $AB = c$, $CB = a = |x_2 - x_1|$, and $CA = b = |y_2 - y_1|$. Then,



$$c^2 = a^2 + b^2$$

$$c^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

$$c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

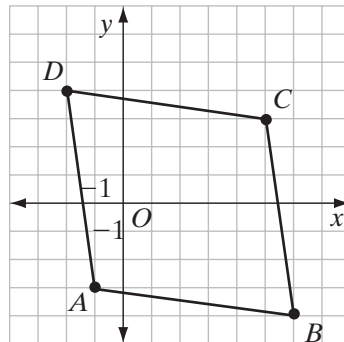
This result is called the **distance formula**. If the endpoints of a line segment in the coordinate plane are $B(x_1, y_1)$ and $A(x_2, y_2)$, then:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLE I

The coordinates of the vertices of quadrilateral $ABCD$ are $A(-1, -3)$, $B(6, -4)$, $C(5, 3)$, and $D(-2, 4)$.

- Prove that $ABCD$ is a rhombus.
- Prove that $ABCD$ is not a square.



Solution

$$\begin{aligned} \text{a. } AB &= \sqrt{(6 - (-1))^2 + (-4 - (-3))^2} \\ &= \sqrt{(7)^2 + (-1)^2} \\ &= \sqrt{49 + 1} \\ &= \sqrt{50} \end{aligned}$$

$$\begin{aligned} BC &= \sqrt{(5 - 6)^2 + (3 - (-4))^2} \\ &= \sqrt{(-1)^2 + (7)^2} \\ &= \sqrt{1 + 49} \\ &= \sqrt{50} \end{aligned}$$

$$\begin{aligned} CD &= \sqrt{(-2 - 5)^2 + (4 - 3)^2} \\ &= \sqrt{(-7)^2 + (1)^2} \\ &= \sqrt{49 + 1} \\ &= \sqrt{50} \end{aligned}$$

$$\begin{aligned} DA &= \sqrt{(-1 - (-2))^2 + (-3 - 4)^2} \\ &= \sqrt{(1)^2 + (-7)^2} \\ &= \sqrt{1 + 49} \\ &= \sqrt{50} \end{aligned}$$

The lengths of the sides of the quadrilateral are equal. Therefore, the quadrilateral is a rhombus.

- b. If a rhombus is a square, then it has a right angle.

METHOD 1

$$\begin{aligned} AC &= \sqrt{(5 - (-1))^2 + (3 - (-3))^2} \\ &= \sqrt{(6)^2 + (6)^2} \\ &= \sqrt{36 + 36} \\ &= \sqrt{72} \end{aligned}$$

If $\angle B$ is a right angle, then

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ (\sqrt{72})^2 &\stackrel{?}{=} (\sqrt{50})^2 + (\sqrt{50})^2 \\ 72 &\neq 50 + 50 \quad \times \end{aligned}$$

Therefore, $\triangle ABC$ is not a right triangle, $\angle B$ is not a right angle and $ABCD$ is not a square.

METHOD 2

$$\begin{aligned} \text{slope of } \overline{AB} &= \frac{-4 - (-3)}{6 - (-1)} \\ &= \frac{-1}{7} \\ &= -\frac{1}{7} \end{aligned}$$

$$\begin{aligned} \text{slope of } \overline{BC} &= \frac{3 - (-4)}{5 - 6} \\ &= \frac{7}{-1} \\ &= -7 \end{aligned}$$

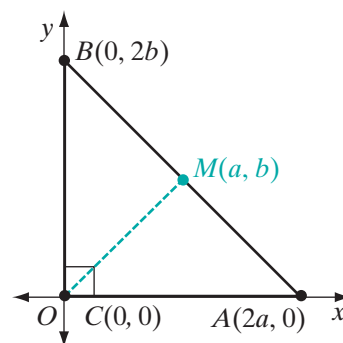
The slope of $\overline{AB}$ is not equal to the negative reciprocal of the slope of $\overline{BC}$. Therefore, $\overline{AB}$ is not perpendicular to $\overline{BC}$, $\angle B$ is not a right angle and the rhombus is not a square. ■

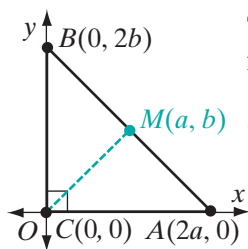
EXAMPLE 2

Prove that the midpoint of the hypotenuse of a right triangle is equidistant from the vertices using the distance formula.

Proof We will use a coordinate proof. The triangle can be placed at any convenient position. Let right triangle ABC have vertices $A(2a, 0)$, $B(0, 2b)$, and $C(0, 0)$. Let M be the midpoint of the hypotenuse $\overline{AB}$. The coordinates of M , the midpoint of $\overline{AB}$, are

$$\left(\frac{2a + 0}{2}, \frac{0 + 2b}{2} \right) = (a, b)$$





Then, since M is the midpoint of $\overline{AB}$, $AM = BM$, and using the distance formula:

$$\begin{aligned} AM &= BM = \sqrt{(a-0)^2 + (b-2b)^2} \\ &= \sqrt{a^2 + (-b)^2} \\ &= \sqrt{a^2 + b^2} \end{aligned}$$

$$\begin{aligned} CM &= \sqrt{(a-0)^2 + (b-0)^2} \\ &= \sqrt{a^2 + b^2} \end{aligned}$$

Therefore, the midpoint of the hypotenuse is equidistant from the vertices of the triangle. ■

EXAMPLE 3

Prove that the medians to the base angles of an isosceles triangle are congruent.

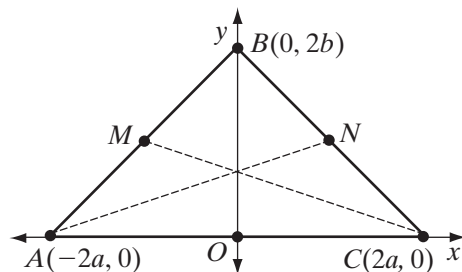
Given: Isosceles $\triangle ABC$ with vertices $A(-2a, 0)$, $B(0, 2b)$, $C(2a, 0)$.
Let M be the midpoint of $\overline{AB}$
and N be the midpoint of $\overline{BC}$.

Prove: $\overline{CM} \cong \overline{AN}$

Proof The coordinates of M are
 $\left(\frac{-2a+0}{2}, \frac{0+2b}{2}\right) = (-a, b)$.

$$\begin{aligned} \text{The length of } \overline{CM} &\text{ is} \\ \sqrt{(-a-2a)^2 + (b-0)^2} \\ &= \sqrt{(-3a)^2 + b^2} \\ &= \sqrt{9a^2 + b^2} \end{aligned}$$

$CM = AN$; therefore, $\overline{CM} \cong \overline{AN}$. ■



The coordinates of N are
 $\left(\frac{2a+0}{2}, \frac{0+2b}{2}\right) = (a, b)$.

$$\begin{aligned} \text{The length of } \overline{AN} &\text{ is} \\ \sqrt{(-2a-a)^2 + (0-b)^2} \\ &= \sqrt{(-3a)^2 + (-b)^2} \\ &= \sqrt{9a^2 + b^2} \end{aligned}$$

Exercises

Writing About Mathematics

1. Can the distance formula be used to find the length of a line segment when the endpoints of the segment are on the same horizontal line or on the same vertical line? Justify your answer.
2. Explain why $|x_2 - x_1|^2 = (x_2 - x_1)^2$.

Developing Skills

In 3–10, the coordinates of the endpoints of $\overline{AB}$ are given. In each case, find the exact value of AB in simplest form.

3. $A(1, 2), B(4, 6)$
4. $A(-1, -6), B(4, 6)$
5. $A(3, -2), B(5, 4)$
6. $A(0, 2), B(3, -1)$
7. $A(1, 2), B(3, 4)$
8. $A(-5, 2), B(1, -6)$
9. $A(6, 2), B(1, -3)$
10. $A(-3, 3), B(3, -3)$
11. The coordinates of A are $(0, 4)$ and the x -coordinate of B is 5. What is the y -coordinate of B if $AB = 13$? (Two answers are possible.)
12. The coordinates of M are $(2, -1)$ and the y -coordinate of N is 5. What is the x -coordinate of N if $MN = 3\sqrt{5}$? (Two answers are possible.)
13. The vertices of a quadrilateral are $A(0, -2), B(5, -2), C(8, 2), D(3, 2)$. Prove that the quadrilateral is a rhombus using the distance formula.
14. The vertices of a triangle are $P(1, -1), Q(7, 1)$, and $R(3, 3)$.
 - a. Show that $\triangle PQR$ is an isosceles triangle.
 - b. Show that $\triangle PQR$ is a right triangle using the Pythagorean Theorem.
 - c. Show that the midpoint of the hypotenuse is equidistant from the vertices.
15. The vertices of a triangle are $L(1, -1), M(7, -3)$, and $N(2, 2)$.
 - a. Show that $\triangle LMN$ is a scalene triangle.
 - b. Show that $\triangle LMN$ is a right triangle using the Pythagorean Theorem.
 - c. Show that the midpoint of $\overline{MN}$ is equidistant from the vertices.
16. The vertices of $\triangle DEF$ are $D(-2, -3), E(5, 0)$, and $F(-2, 3)$. Show that $\overline{DE} \cong \overline{FE}$.
17. The coordinates of the vertices of $\triangle BAT$ are $B(-2, 7), A(2, -1)$, and $T(11, -4)$.
 - a. Find the equation of the line that is the altitude from B to $\overline{AT}$.
 - b. Find the coordinates of D , the foot of the perpendicular or the **foot of the altitude** from part a.
 - c. Find the length of the altitude $\overline{BD}$.
18. The coordinates of the vertices of $\triangle EDF$ are $E(-2, 0), D(4, 0)$, and $F(1, 3\sqrt{3})$.
 - a. Find ED, DF , and FE .
 - b. Is $\triangle EDF$ equilateral? Justify your answer.
19. The vertices of $\triangle ABC$ are $A(1, -1), B(4, 1)$, and $C(2, 4)$.
 - a. Find the coordinates of the vertices of $\triangle A'B'C'$, the image of $\triangle ABC$ under the transformation D_2 .
 - b. Show that distance is *not* preserved under the dilation.

- c. Show that $\triangle A'B'C' \sim \triangle ABC$ using SSS $\sim$.
 - d. Use part c to show that the angle measures of $\triangle ABC$ are preserved under the dilation.
20. The vertices of quadrilateral $ABCD$ are $A(2, 0)$, $B(3, -1)$, $C(4, 1)$, and $D(3, 2)$.
- a. Show that $ABCD$ is a parallelogram using the distance formula.
 - b. Find the coordinates of the vertices of quadrilateral $A'B'C'D'$, the image of $ABCD$ under the transformation D_3 .
 - c. Show that $A'B'C'D'$ is a parallelogram using the distance formula.
 - d. Use part c to show that the images of the parallel segments of $ABCD$ are also parallel under the dilation.
21. The vertices of quadrilateral $ABCD$ are $A(-2, -2)$, $B(2, 0)$, $C(3, 3)$, and $D(-1, 1)$. Use the distance formula to prove that $ABCD$ is a parallelogram but not a rhombus.
22. The vertices of $\triangle ABC$ are $A(0, -2)$, $B(4, 6)$, and $C(-2, 4)$. Prove that $\triangle ABC$ is an isosceles right triangle using the Pythagorean Theorem.

Applying Skills

23. Use the distance formula to prove that $(-a, 0)$, $(0, b)$ and $(a, 0)$ are the vertices of an isosceles triangle.
24. The vertices of square $EFGH$ are $E(0, 0)$, $F(a, 0)$, $G(a, a)$, and $H(0, a)$. Prove that the diagonals of a square, $\overline{EG}$ and $\overline{FH}$, are congruent and perpendicular.
25. The vertices of quadrilateral $ABCD$ are $A(0, 0)$, $B(b, c)$, $C(b + a, c)$, and $D(a, 0)$. Prove that if $a^2 = b^2 + c^2$ then $ABCD$ is a rhombus.
26. Prove the midpoint formula using the distance formula. Let P and Q have coordinates (x_1, y_1) and (x_2, y_2) , respectively. Let M have coordinates $(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2})$. M is the midpoint of $\overline{PQ}$ if and only if $PM = MQ$.
- a. Find PM .
 - b. Find MQ .
 - c. Show that $PM = MQ$.
27. The vertices of $\overline{WX}$ are $W(w, y)$ and $X(x, z)$.
- a. What are the coordinates of $\overline{W'X'}$, the image of $\overline{WX}$ under the dilation D_k ?
 - b. Show, using the distance formula, that $W'X'$ is k times the length of WX .

CHAPTER SUMMARY*Definitions
to Know*

- The **ratio of two numbers**, a and b , where b is not zero, is the number $\frac{a}{b}$.
- A **proportion** is an equation that states that two ratios are equal.
- In the proportion $\frac{a}{b} = \frac{c}{d}$, the first and fourth terms, a and d , are the **extremes** of the proportion, and the second and third terms, b and c , are the **means**.
- If the means of a proportion are equal, the **mean proportional** is one of the means.
- Two line segments are **divided proportionally** when the ratio of the lengths of the parts of one segment is equal to the ratio of the lengths of the parts of the other.
- Two polygons are **similar** if there is a one-to-one correspondence between their vertices such that:
 1. All pairs of corresponding angles are congruent.
 2. The ratios of the lengths of all pairs of corresponding sides are equal.
- The **ratio of similitude** of two similar polygons is the ratio of the lengths of corresponding sides.
- A **dilation** of k is a transformation of the plane such that:
 1. The image of point O , the center of dilation, is O .
 2. When k is positive and the image of P is P' , then $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ are the same ray and $OP' = kOP$.
 3. When k is negative and the image of P is P' , then $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ are opposite rays and $OP' = -kOP$.
- The **projection of a point on a line** is the foot of the perpendicular drawn from that point to the line.
- The **projection of a segment on a line**, when the segment is not perpendicular to the line, is the segment whose endpoints are the projections of the endpoints of the given line segment on the line.
- A **Pythagorean triple** is a set of three integers that can be the lengths of the sides of a right triangle.

Postulates

- 12.1** Any geometric figure is similar to itself. (Reflexive property)
- 12.2** A similarity between two geometric figures may be expressed in either order. (Symmetric property)
- 12.3** Two geometric figures similar to the same geometric figure are similar to each other. (Transitive property)
- 12.4** For any given triangle there exist a similar triangle with any given ratio of similitude.

- Theorems**
- 12.1** In a proportion, the product of the means is equal to the product of the extremes.
 - 12.1a** In a proportion, the means may be interchanged.
 - 12.1b** In a proportion, the extremes may be interchanged.
 - 12.1c** If the products of two pairs of factors are equal, the factors of one pair can be the means and the factors of the other the extremes of a proportion.
 - 12.2** A line segment joining the midpoints of two sides of a triangle is parallel to the third side and its length is one-half the length of the third side.
 - 12.3** Two line segments are divided proportionally if and only if the ratio of the length of a part of one segment to the length of the whole is equal to the ratio of the corresponding lengths of the other segment.
 - 12.4** Two triangles are similar if two angles of one triangle are congruent to two corresponding angles of the other. (AA~)
 - 12.5** Two triangles are similar if the three ratios of corresponding sides are equal. (SSS~)
 - 12.6** Two triangles are similar if the ratios of two pairs of corresponding sides are equal and the corresponding angles included between these sides are congruent. (SAS~)
 - 12.7** A line is parallel to one side of a triangle and intersects the other two sides if and only if the points of intersection divide the sides proportionally.
 - 12.8** Under a dilation, angle measure is preserved.
 - 12.9** Under a dilation, midpoint is preserved.
 - 12.10** Under a dilation, collinearity is preserved.
 - 12.11** If two triangles are similar, the lengths of corresponding altitudes have the same ratio as the lengths of any two corresponding sides.
 - 12.12** If two triangles are similar, the lengths of corresponding medians have the same ratio as the lengths of any two corresponding sides.
 - 12.13** If two triangles are similar, the lengths of corresponding angle bisectors have the same ratio as the lengths of any two corresponding sides.
 - 12.14** Any two medians of a triangle intersect in a point that divides each median in the ratio 2 : 1.
 - 12.15** The medians of a triangle are concurrent.
 - 12.16** The altitude to the hypotenuse of a right triangle divides the triangle into two triangles that are similar to each other and to the original triangle.
 - 12.16a** The length of each leg of a right triangle is the mean proportional between the length of the projection of that leg on the hypotenuse and the length of the hypotenuse.
 - 12.16b** The length of the altitude to the hypotenuse of a right triangle is the mean proportional between the lengths of the projections of the legs on the hypotenuse.
 - 12.17** A triangle is a right triangle if and only if the square of the length of the longest side is equal to the sum of the squares of the lengths of the other two sides.

Formulas In the coordinate plane, under a dilation of k with the center at the origin:

$$P(x, y) \rightarrow P'(kx, ky) \quad \text{or} \quad D_k(x, y) = (kx, ky)$$

If x is the length of a leg of an isosceles right triangle and s is the length of the hypotenuse, then:

$$x = \frac{\sqrt{2}}{2}s \quad \text{and} \quad s = x\sqrt{2}$$

If s is the length of a side of an equilateral triangle and h is the length of an altitude then:

$$h = \frac{\sqrt{3}}{2}s$$

If the endpoints of a line segment in the coordinate plane are $B(x_1, y_1)$ and $A(x_2, y_2)$, then:

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

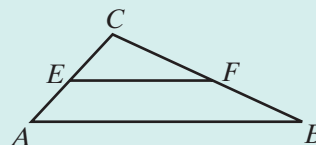
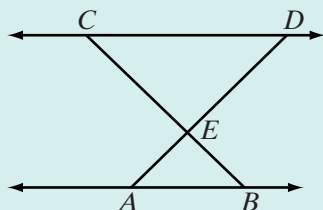
VOCABULARY

- 12-1** Similar • Ratio of two numbers • Proportion • Extremes • Means • Mean proportional • Geometric mean
- 12-2** Midsegment theorem • Line segments divided proportionally
- 12-3** Similar polygons • Ratio of similitude • Constant of proportionality
- 12-4** Postulate of similarity • AA triangle similarity • SSS similarity theorem • SAS similarity theorem
- 12-5** Dilation • Enlargement • Contraction • Constant of dilation, k
- 12-7** Centroid
- 12-8** Projection of a point on a line • Projection of a segment on a line
- 12-9** Pythagorean Theorem • Pythagorean triple • 45-45-degree right triangle • 30-60-degree right triangle
- 12-10** Distance formula • Foot of an altitude

REVIEW EXERCISES

1. Two triangles are similar. The lengths of the sides of the smaller triangle are 5, 6, and 9. The perimeter of the larger triangle is 50. What are the lengths of the sides of the larger triangle?
2. The measure of one angle of right $\triangle ABC$ is 67° and the measure of an angle of right $\triangle LMN$ is 23° . Are the triangles similar? Justify your answer.

3. A line parallel to side $\overline{AB}$ of $\triangle ABC$ intersects $\overline{AC}$ at E and $\overline{BC}$ at F . If $EC = 12$, $AC = 20$, and $AB = 15$, find EF .
4. A line intersects side $\overline{AC}$ of $\triangle ABC$ at E and $\overline{BC}$ at F . If $EC = 4$, $AC = 12$, $FC = 5$, $BC = 15$, prove that $\triangle EFC \sim \triangle ABC$.
5. The altitude to the hypotenuse of a right triangle divides the hypotenuse into two segments. If the length of the altitude is 12 and the length of the longer segment is 18, what is the length of the shorter segment?
6. In $\triangle LMN$, $\angle L$ is a right angle, $\overline{LP}$ is an altitude, $MP = 8$, and $PN = 32$.
 - a. Find LP .
 - b. Find MN .
 - c. Find ML .
 - d. Find NL .
7. The length of a side of an equilateral triangle is 18 centimeters. Find, to the nearest tenth of a centimeter, the length of the altitude of the triangle.
8. The length of the altitude to the base of an isosceles triangle is 10.0 centimeters and the length of the base is 14.0 centimeters. Find, to the nearest tenth of a centimeter, the length of each of the legs.
9. The coordinates of the endpoints of $\overline{PQ}$ are $P(2, 7)$ and $Q(8, -1)$.
 - a. Find the coordinates of the endpoints of $\overline{P'Q'}$ under the composition $D_2 \circ r_{x\text{-axis}}$.
 - b. What is the ratio $\frac{PQ}{P'Q'}$?
 - c. What are the coordinates of M , the midpoint of $\overline{PQ}$?
 - d. What are the coordinates of M' , the image of M under $D_2 \circ r_{x\text{-axis}}$?
 - e. What are the coordinates of N , the midpoint of $\overline{P'Q'}$?
 - f. Is M' the midpoint of $\overline{P'Q'}$? Justify your answer.
10. If $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$ and $\overline{AD}$ and $\overline{BC}$ intersect at E , prove that $\triangle ABE \sim \triangle DCE$.
11. A line intersects $\overline{AC}$ at E and $\overline{BC}$ at F . If $\triangle ABC \sim \triangle EFC$, prove that $\overleftrightarrow{EF} \parallel \overleftrightarrow{AB}$.



12. Find the length of the altitude to the bases of isosceles trapezoid $KLMN$ if $KL = 20$ cm, $MN = 38$ cm, and $KN = 15$ cm.
13. Find the length of a side of a rhombus if the measures of the diagonals of the rhombus are 30 inches and 40 inches.

14. The length of a side of a rhombus is 26.0 centimeters and the length of one diagonal is 28.0 centimeters. Find to the nearest tenth the length of the other diagonal.
15. The coordinates of the vertices of $\triangle RST$ are $R(4, 1)$, $S(-3, 2)$, and $T(-2, -1)$.
 - a. Find the length of each side of the triangle in simplest radical form.
 - b. Prove that the triangle is a right triangle.
16. The vertices of $\triangle ABC$ are $A(0, 0)$, $B(4, 3)$, and $C(0, 5)$.
 - a. Prove that $\triangle ABC$ is isosceles.
 - b. The median to $\overline{BC}$ is $\overline{AD}$. Find the coordinates of D .
 - c. Find AD and DB .
 - d. Prove that $\overline{AD}$ is the altitude to $\overline{BC}$ using the Pythagorean Theorem.
17. The vertices of $\triangle ABC$ are $A(-2, 1)$, $B(2, -1)$, and $C(0, 3)$.
 - a. Find the coordinates of $\triangle A'B'C'$, the image of $\triangle ABC$ under D_3 .
 - b. Find, in radical form, the lengths of the sides of $\triangle ABC$ and of $\triangle A'B'C'$.
 - c. Prove that $\triangle ABC \sim \triangle A'B'C'$.
 - d. Find the coordinates of P , the centroid of $\triangle ABC$.
 - e. Let M be the midpoint of $\overline{AB}$. Prove that $\frac{CP}{PM} = \frac{2}{1}$.
 - f. Find the coordinates of P' , the centroid of $\triangle A'B'C'$.
 - g. Is P' the image of P under D_3 ?
 - h. Let M' be the midpoint of $\overline{A'B'}$. Prove that $\frac{C'P'}{P'M'} = \frac{2}{1}$.
18. A right circular cone is cut by a plane through a diameter of the base and the vertex of the cone. If the diameter of the base is 20 centimeters and the height of the cone is 24 centimeters, what is the slant height of the cone?

Exploration

A line parallel to the shorter sides of a rectangle can divide the rectangle into a square and a smaller rectangle. If the smaller rectangle is similar to the given rectangle, then both rectangles are called **golden rectangles** and the ratio of the lengths of their sides is called the **golden ratio**. The golden ratio is $(1 + \sqrt{5}) : 2$.



STEP 1. Draw square $ABCD$.

STEP 3. Draw the ray $\overrightarrow{AB}$.

STEP 5. Draw the ray $\overrightarrow{DC}$.

a. Let $AB = BC = 2$ and $EB = 1$. Find $EC = EF$, $AF = AE + EF$, and $BF = EF - EB$. Express each length as an irrational number in simplest radical form.

c. Repeat steps 1 through 6 using a different square. Let $AB = x$. Complete parts **a** and **b**. Do you obtain the same ratio?

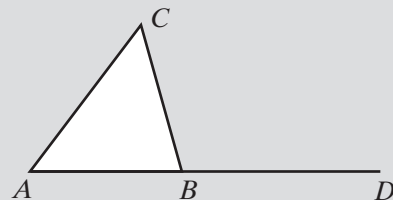
The diagram shows a rectangle $ABCD$ with vertices A (bottom-left), B (bottom-right), C (top-right), and D (top-left). The side AB is labeled with a teal '1'. A dashed teal arc of radius 1 is centered at C . This arc intersects the horizontal line containing AB at point F (to the right of B) and the vertical line containing BC at point G (above C). A right-angle symbol is shown at point F between the horizontal line ABF and the vertical line FG .

Chapters 1–12

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

- The length and width of a rectangle are 16 centimeters and 12 centimeters. What is the length of a diagonal of the rectangle?
(1) 4 cm (2) 20 cm (3) 25 cm (4) $4\sqrt{7}$ cm
- The measure of an angle is 12 degrees more than twice the measure of its supplement. What is the measure of the angle?
(1) 26 (2) 56 (3) 64 (4) 124
- What is the slope of the line through $A(2, 8)$ and $B(-4, -1)$?
(1) $-\frac{3}{2}$ (2) $\frac{3}{2}$ (3) $-\frac{2}{3}$ (4) $\frac{2}{3}$

4. The measures of the opposite angles of a parallelogram are represented by $2x + 34$ and $3x - 12$. Find the value of x .
 (1) 22 (2) 46 (3) 78 (4) 126
5. Which of the following do not determine a plane?
 (1) three lines each perpendicular to the other two
 (2) two parallel lines (3) two intersecting lines
 (4) a line and a point not on it
6. Which of the following cannot be the measures of the sides of a triangle?
 (1) 5, 7, 8 (2) 3, 8, 9 (3) 5, 7, 7 (4) 2, 6, 8
7. Under a rotation of 90° about the origin, the image of the point whose coordinates are $(3, -2)$ is the point whose coordinates are
 (1) $(2, -3)$ (2) $(2, 3)$ (3) $(-3, 2)$ (4) $(-2, -3)$
8. If a conditional statement is true, which of the following must be true?
 (1) converse (3) contrapositive
 (2) inverse (4) biconditional
9. In the figure, side $\overline{AB}$ of $\triangle ABC$ is extended through B to D . If $m\angle CBD = 105$ and $m\angle A = 53$, what is the measure of $\angle C$?
 (1) 22 (3) 75
 (2) 52 (4) 158



10. A parallelogram with one right angle must be
 (1) a square.
 (2) a rectangle.
 (3) a rhombus.
 (4) a trapezoid.

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. *Given:* $\overline{ABCD}$ with $\overline{AB} \cong \overline{CD}$, E not on $\overleftrightarrow{ABCD}$, and $\overline{BE} \cong \overline{CE}$.
Prove: $\overline{AE} \cong \overline{DE}$.
12. *Given:* $\overline{RS}$ perpendicular to plane p at R , points A and B in plane p , and $\overline{RA} \cong \overline{RB}$.
Prove: $\overline{SA} \cong \overline{SB}$

Part III

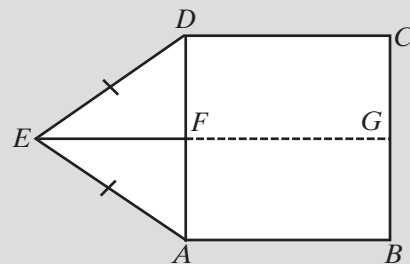
Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. The radius of the base of a right circular cone is one-half the slant height of the cone. The radius of the base is 2.50 feet.
- Find the lateral area of the cone to the nearest tenth of a square foot.
 - Find the volume of the cone to the nearest tenth of a cubic foot.
14. The coordinates of the vertices of $\triangle ABC$ are $A(-1, 0)$, $B(4, 0)$, and $C(2, 6)$.
- Write an equation of the line that contains the altitude from C to $\overline{AB}$.
 - Write an equation of the line that contains the altitude from B to $\overline{AC}$.
 - What are the coordinates of D , the intersection of the altitudes from C and from B ?
 - Write an equation for $\overleftrightarrow{AD}$.
 - Is $\overleftrightarrow{AD}$ perpendicular to $\overline{BC}$, that is, does $\overleftrightarrow{AD}$ contain the altitude from A to $\overline{BC}$?

Part IV

Answer all questions in this part. Each correct answer will receive 6 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. In the diagram, $ABCD$ is a rectangle and $\triangle ADE$ is an isosceles triangle with $\overline{AE} \cong \overline{DE}$. If $\overline{EF}$, a median to $\overline{AD}$ of $\triangle ADE$, is extended to intersect $\overline{BC}$ at G , prove that G is the midpoint of $\overline{BC}$.



16. The coordinates of the vertices of $\triangle ABC$ are $A(2, 5)$, $B(3, 1)$, and $C(6, 4)$.
- Find the coordinates of $\triangle A'B'C'$, the image of $\triangle ABC$ under the composition $r_{y=x} \circ r_{x\text{-axis}}$.
 - Show that $r_{y=x} \circ r_{x\text{-axis}}(x, y) = R_{90^\circ}(x, y)$.