CHAPTER

11

SPECIAL PRODUCTS AND FACTORS

The owners of a fruit farm intend to extend their orchards by planting 100 new apple trees. The trees are to be planted in rows, with the same number of trees in each row. In order to decide how the trees are to be planted, the owners will use whole numbers that are factors of 100 to determine the possible arrangements:

1 row of 100 trees	20 rows of 5 trees each
2 rows of 50 trees each	25 rows of 4 trees each
4 rows of 25 trees each	50 rows of 2 trees each
5 rows of 20 trees each	100 rows of 1 tree each
10 rows of 10 trees each	

From this list of possibilities, the arrangement that best fits the dimensions of the land to be planted can be chosen.

If the owner had intended to plant 90 trees, how many possible arrangements would there be?

From earliest times, the study of factors and prime numbers has fascinated mathematicians, leading to the discovery of many important principles. In this chapter you will extend your knowledge of the factors of whole numbers, study the products of special binomials, and learn to write polynomials in factored form.

11-1 FACTORS AND FACTORING

When two numbers are multiplied, the result is called their **product**. The numbers that are multiplied are *factors* of the product. Since $3(5) = 15$, the numbers 3 and 5 are factors of 15.

Factoring a number is the process of finding those numbers whose product is the given number. Usually, when we factor, we are finding the factors of an integer and we find only those factors that are integers. We call this **factoring over the set of integers**.

Factors of a product can be found by using division. Over the set of integers, if the divisor and the quotient are both integers, then they are factors of the dividend. For example, $35 \div 5 = 7$. Thus, $35 = 5(7)$, and 5 and 7 are factors of 35.

Every positive integer that is the product of two positive integers is also the product of the opposites of those integers.

$$+21 = (+3)(+7) \qquad\qquad +21 = (-3)(-7)$$

Every negative integer that is the product of a positive integer and a negative integer is also the product of the opposites of those integers.

$$-21 = (+3)(-7) \qquad\qquad -21 = (-3)(+7)$$

Usually, when we factor a positive integer, we write only the positive integral factors.

Two factors of any number are 1 and the number itself. To find other integral factors, if they exist, we use division, as stated above. We let the number being factored be the dividend, and we divide this number in turn by the whole numbers 2, 3, 4, and so on. If the quotient is an integer, then both the divisor and the quotient are factors of the dividend.

For example, use a calculator to find the integral factors of 126. We will use integers as divisors and look for quotients that are integers. Pairs of factors of 126 are listed to the right of the quotients.

Quotients	Pairs of Factors of 126	Quotients	Pairs of Factors of 126
$126 \div 1 = 126$	$1 \cdot 126$	$126 \div 7 = 18$	$7 \cdot 18$
$126 \div 2 = 63$	$2 \cdot 63$	$126 \div 8 = 15.75$	—
$126 \div 3 = 42$	$3 \cdot 42$	$126 \div 9 = 14$	$9 \cdot 14$
$126 \div 4 = 31.5$	—	$126 \div 10 = 12.6$	—
$126 \div 5 = 25.2$	—	$126 \div 11 = 11.\overline{45}$	—
$126 \div 6 = 21$	$6 \cdot 21$	$126 \div 12 = 10.5$	—

When the quotient is smaller than the divisor (here, $10.5 < 12$), we have found all possible positive integral factors.

The factors of 126 are 1, 2, 3, 6, 7, 9, 14, 18, 21, 42, 63, and 126.

Recall that a *prime number* is an integer greater than 1 that has no positive integral factors other than itself and 1. The first seven prime numbers are 2, 3, 5, 7, 11, 13, and 17. Integers greater than 1 that are not prime are called *composite numbers*.

In general, a positive integer greater than 1 is a prime or can be expressed as the product of prime factors. Although the factors may be written in any order, there is one and only one combination of prime factors whose product is a given composite number. As shown below, a prime factor may occur in the product more than once.

$$21 = 3 \cdot 7$$
$$20 = 2 \cdot 2 \cdot 5 \text{ or } 2^2 \cdot 5$$

To express a positive integer, for example 280, as the product of primes, we start with any pair of positive integers, say 28 and 10, whose product is the given number. Then, we factor these factors and continue to factor the factors until all are primes. Finally, we rearrange these factors in numerical order, as shown at the right.

$$280$$
$$280 = \quad 28 \quad \cdot \quad 10$$
$$280 = \quad 2 \cdot 14 \cdot 2 \cdot 5$$
$$280 = 2 \cdot 2 \cdot 7 \cdot 2 \cdot 5$$
$$\text{or}$$
$$280 = \quad 2^3 \cdot 5 \cdot 7$$

Expressing each of two integers as the product of prime factors makes it possible to discover the greatest integer that is a factor of both of them. We call this factor the **greatest common factor (GCF)** of these integers.

Let us find the greatest common factor of 180 and 54.

$$180 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \quad \text{or} \quad 2^2 \cdot 3^2 \cdot 5$$
$$\downarrow \quad\quad \downarrow \ \downarrow$$
$$54 = 2 \cdot 3 \cdot 3 \cdot 3 \quad\quad \text{or} \quad 2 \cdot 3^3$$
$$\downarrow \quad\quad \downarrow \ \downarrow$$
$$\text{Greatest common factor} = 2 \cdot \quad\ 3 \cdot 3 \quad \text{or} \quad 2 \cdot 3^2 \quad \text{or} \quad 18$$

Only the prime numbers 2 and 3 are factors of both 180 and 54. We see that the greatest number of times that 2 appears as a factor of both 180 and 54 is once; the greatest number of times that 3 appears as a factor of both 180 and 54 is twice. Therefore, the greatest common factor of 180 and 54 is $2 \cdot 3 \cdot 3$, or $2 \cdot 3^2$, or 18.

To find the greatest common factor of two or more monomials, find the product of the numerical and variable factors that are common to the monomials. For example, let us find the greatest common factor of $24a^3b^2$ and $18a^2b$.

$$24a^3b^2 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot \quad a \cdot a \cdot a \cdot b \cdot b$$
$$\downarrow \quad\quad\quad \downarrow \quad \downarrow \ \downarrow \quad\quad \downarrow$$
$$18a^2b = 2 \cdot \quad\quad 3 \cdot 3 \cdot a \cdot a \cdot \quad b$$
$$\downarrow \quad\quad\quad \downarrow \quad \downarrow \ \downarrow \quad\quad \downarrow$$
$$\text{Greatest common factor} = 2 \cdot \quad\quad 3 \cdot \quad a \cdot a \cdot \quad b = 6a^2b$$

The greatest common factor of $24a^3b^2$ and $18a^2b$ is $6a^2b$.

When we are expressing an algebraic factor, such as $6a^2b$, we will agree that:

- Numerical coefficients need not be factored. (6 need not be written as $2 \cdot 3$.)
- Powers of variables need not be represented as the product of several equal factors. (a^2b need not be written as $a \cdot a \cdot b$.)

EXAMPLE 1

Write all positive integral factors of 72.

Solution

Quotients	Pairs of Factors of 72	Quotients	Pairs of Factors of 72
$72 \div 1 = 72$	$1 \cdot 72$	$72 \div 5 = 14.4$	—
$72 \div 2 = 36$	$2 \cdot 36$	$72 \div 6 = 12$	$6 \cdot 12$
$72 \div 3 = 24$	$3 \cdot 24$	$72 \div 7 = 10.\overline{285714}$	—
$72 \div 4 = 18$	$4 \cdot 18$	$72 \div 8 = 9$	$8 \cdot 9$

It is not necessary to divide by 9, since 9 has just been listed as a factor.

Answer The factors of 72 are 1, 2, 3, 4, 6, 8, 9, 12, 18, 24, 36, and 72. ▪

EXAMPLE 2

Express 700 as a product of prime factors.

Solution
$$700 = 2 \cdot 350$$
$$700 = 2 \cdot 2 \cdot 175$$
$$700 = 2 \cdot 2 \cdot 5 \cdot 35$$
$$700 = 2 \cdot 2 \cdot 5 \cdot 5 \cdot 7 \text{ or } 2^2 \cdot 5^2 \cdot 7$$

Answer $2^2 \cdot 5^2 \cdot 7$ ▪

EXAMPLE 3

Find the greatest common factor of the monomials $60r^2s^4$ and $36rs^2t$.

Solution To find the greatest common factor of two monomials, write each as the product of primes and variables to the first power and choose all factors that occur in each product.

$$60r^2s^4 = 2 \cdot 2 \cdot 3 \cdot 5 \cdot r \cdot r \cdot s \cdot s \cdot s \cdot s$$
$$36rs^2t = 2 \cdot 2 \cdot 3 \cdot 3 \cdot r \cdot s \cdot s \cdot t$$

Greatest common factor $= 2 \cdot 2 \cdot 3 \cdot r \cdot s \cdot s$

Answer $12rs^2$ ▪

EXERCISES

Writing About Mathematics

1. Ross said that some pairs of number such as 5 and 12 have no greatest common factor. Do you agree with Ross? Explain why or why not.

2. To find the factors of 200, Chaz tried all of the integers from 2 to 14. Should Chaz have used more numbers? Explain why or why not.

Developing Skills

In 3–12, tell whether each integer is prime, composite, or neither.

3. 5	**4.** 8	**5.** 13	**6.** 18	**7.** 73
8. 36	**9.** 41	**10.** 49	**11.** 57	**12.** 1

In 13–22, express each integer as a product of prime numbers.

13. 35	**14.** 18	**15.** 144	**16.** 77	**17.** 128
18. 400	**19.** 202	**20.** 129	**21.** 590	**22.** 316

In 23–34, write all the positive integral factors of each given number.

23. 26	**24.** 50	**25.** 36	**26.** 88	**27.** 100	**28.** 242
29. 37	**30.** 62	**31.** 253	**32.** 102	**33.** 70	**34.** 169

35. The product of two monomials is $36x^3y^4$. Find the second factor if the first factor is:
a. $3x^2y^3$ **b.** $6x^3y^2$ **c.** $12xy^4$ **d.** $9x^3y$ **e.** $18x^3y^2$

36. The product of two monomials is $81c^9d^{12}e$. Find the second factor if the first factor is:
a. $27e$ **b.** c^2d^{11} **c.** $3cd^3e$ **d.** $9c^4d^4$ **e.** $81c^5d^9e$

In 37–42, find, in each case, the greatest common factor of the given integers.

37. 10; 15	**38.** 12; 28	**39.** 14; 35
40. 18; 24; 36	**41.** 75; 50	**42.** 72; 108

In 43–51, find, in each case, the greatest common factor of the given monomials.

43. $4x$; $4y$	**44.** $4r$; $6r^2$	**45.** $8xy$; $6xz$
46. $10x^2$; $15xy^2$	**47.** $36xy^2z$; $-27xy^2z^2$	**48.** $24ab^2c^3$; $18ac^2$
49. $14a^2b$; $13ab$	**50.** $36xyz$; $25xyz$	**51.** $2ab^2c$; $3x^2yz$

11-2 COMMON MONOMIAL FACTORS

To **factor a polynomial** over the set of integers means to express the given polynomial as the product of polynomials whose coefficients are integers. For example, since $2(x + y) = 2x + 2y$, the polynomial $2x + 2y$ can be written in factored form as $2(x + y)$. The monomial 2 is a factor of each term of the polynomial $2x + 2y$. Therefore, 2 is called a **common monomial factor** of the polynomial $2x + 2y$.

To factor a polynomial, we look first for the **greatest common monomial factor**, that is, the greatest monomial that is a factor of each term of the polynomial.

For example:

1. Factor $4rs + 8st$. There are many common factors of $4rs$ and $8st$ such as $2, 4, 2s,$ and $4s$. The greatest common monomial factor is $4s$. We divide $4rs + 8st$ by $4s$ to obtain the quotient $r + 2t$, which is the second factor. Therefore, the polynomial $4rs + 8st = 4s(r + 2t)$.

2. Factor $3x + 4y$. We notice that 1 is the only common factor of $3x$ and $4y$, so the second factor is $3x + 4y$. We say that $3x + 4y$ is a **prime polynomial**. A polynomial with integers as coefficients is a prime polynomial if its only factors are 1 and the polynomial itself.

Procedure

To factor a polynomial whose terms have a common monomial factor:

1. Find the greatest monomial that is a factor of each term of the polynomial.

2. Divide the polynomial by the factor found in step 1. The quotient is the other factor.

3. Express the polynomial as the product of the two factors.

We can check by multiplying the factors to obtain the original polynomial.

EXAMPLE 1

Write in factored form: $6c^3d - 12c^2d^2 + 3cd$.

Solution (1) $3cd$ is the greatest common factor of $6c^3d, -12c^2d^2,$ and $3cd$.

(2) To find the other factor, divide $6c^3d - 12c^2d^2 + 3cd$ by $3cd$.

$$(6c^3d - 12c^2d^2 + 3cd) \div 3cd = \frac{6c^3d}{3cd} - \frac{12c^2d^2}{3cd} + \frac{3cd}{3cd}$$
$$= 2c^2 - 4cd + 1$$

Answer $3cd(2c^2 - 4cd + 1)$

EXERCISES

Writing About Mathematics

1. Ramón said that the factored form of $3a + 6a^2 + 15a^3$ is $3a(2a + 5a^2)$. Do you agree with Ramón? Explain why or why not.

2. a. The binomial $12a^2 + 20ab$ can be written as $4a(3a + 5b)$. Does that mean that for all positive integral values of a and b, the value of $12a^2 + 20ab$ has at least two factors other than itself and 1? Justify your answer.

b. The binomial $3a + 5b$ is a prime polynomial. Does that mean that for all positive integral values of a and b, $3a + 5b$ is a prime number? Justify your answer.

Developing Skills

In 3–29, write each expression in factored form.

3. $2a + 2b$	**4.** $3x - 3y$	**5.** $bx + by$
6. $xc - xd$	**7.** $4x + 8y$	**8.** $3m - 6n$
9. $12x - 18y$	**10.** $18c + 27d$	**11.** $8x + 16$
12. $7y - 7$	**13.** $6 - 18c$	**14.** $y^2 - 3y$
15. $2x^2 + 5x$	**16.** $ax - 5ab$	**17.** $3y^4 + 3y^2$
18. $10x - 15x^3$	**19.** $2x - 4x^3$	**20.** $p + prt$
21. $\pi r^2 + \pi rl$	**22.** $\pi r^2 + 2\pi rh$	**23.** $3a^2 - 9$
24. $12y^2 - 4y$	**25.** $3ab^2 - 6a^2b$	**26.** $21r^3s^2 - 14r^2s$
27. $3x^2 - 6x - 30$	**28.** $c^3 - c^2 + 2c$	**29.** $9ab^2 - 6ab - 3a$

Applying Skills

30. The perimeter of a rectangle is represented by $2l + 2w$. Express the perimeter as the product of two factors.

31. The lengths of the parallel sides of a trapezoid are represented by a and b and its height by h. The area of the trapezoid can be written as $\frac{1}{2}ah + \frac{1}{2}bh$. Express this area as the product of two factors.

32. A cylinder is cut from a cube whose edge measures $2s$.

a. Express the volume of the cube in terms of s.

b. If the largest possible cylinder is made, express the volume of the cylinder in terms of s.

c. Use the answers to parts **a** and **b** to express, in terms of s, the volume of the material cut away to form the cylinder.

d. Express the volume of the waste material as the product of two factors.

11-3 THE SQUARE OF A MONOMIAL

To **square a monomial** means to multiply the monomial by itself. For example:

$$(3x)^2 = (3x)(3x) = (3)(3)(x)(x) = (3)^2(x)^2 \text{ or } 9x^2$$
$$(5y^2)^2 = (5y^2)(5y^2) = (5)(5)(y^2)(y^2) = (5)^2(y^2)^2 \text{ or } 25y^4$$
$$(-6b^4)^2 = (-6b^4)(-6b^4) = (-6)(-6)(b^4)(b^4) = (-6)^2(b^4)^2 \text{ or } 36b^8$$
$$(4c^2d^3)^2 = (4c^2d^3)(4c^2d^3) = (4)(4)(c^2)(c^2)(d^3)(d^3) = (4)^2(c^2)^2(d^3)^2 \text{ or } 16c^4d^6$$

When a monomial is a perfect square, its numerical coefficient is a perfect square and the exponent of each variable is an even number. This statement holds true for each of the results shown above.

Procedure

To square a monomial:

1. Find the square of the numerical coefficient.

2. Find the square of each literal factor by multiplying its exponent by 2.

3. The square of the monomial is the product of the expressions found in steps 1 and 2.

EXAMPLE 1

Square each monomial mentally.

	Think	*Write*
a. $(4s^3)^2$	$(4)^2(s^3)^2$	$16s^6$
b. $\left(\frac{2}{5}ab\right)^2$	$\left(\frac{2}{5}\right)^2(a)^2(b)^2$	$\frac{4}{25}a^2b^2$
c. $(-7xy^2)^2$	$(-7)^2(x)^2(y^2)^2$	$49x^2y^4$
d. $(0.3y^2)^2$	$(0.3)^2(y^2)^2$	$0.09y^4$

EXERCISES

Writing About Mathematics

1. Explain why the exponent of each variable in the square of a monomial is an even number.

2. When the square of a number is written in scientific notation, is the exponent of 10 always an even number? Explain why or why not and give examples to justify your answer.

Developing Skills

In 3–22, square each monomial.

3. $(a^2)^2$ **4.** $(b^3)^2$ **5.** $(-d^5)^2$ **6.** $(rs)^2$

7. $(m^2n^2)^2$ **8.** $(-x^3y^2)^2$ **9.** $(3x^2)^2$ **10.** $(-5y^4)^2$

11. $(9ab)^2$ **12.** $(10x^2y^2)^2$ **13.** $(-12cd^3)^2$ **14.** $\left(\frac{3}{4}a\right)^2$

15. $\left(\frac{5}{7}xy\right)^2$ **16.** $\left(-\frac{7}{8}a^2b^2\right)^2$ **17.** $\left(\frac{x}{6}\right)^2$ **18.** $\left(-\frac{4x^2}{5}\right)^2$

19. $(0.8x)^2$ **20.** $(0.5y^2)^2$ **21.** $(0.01xy)^2$ **22.** $(-0.06a^2b)^2$

Applying Skills

In 23–28, each monomial represents the length of a side of a square. Write the monomial that represents the area of the square.

23. $4x$ **24.** $10y$ **25.** $\frac{2}{3}x$ **26.** $1.5x$ **27.** $3x^2$ **28.** $4x^2y^3$

11-4 MULTIPLYING THE SUM AND THE DIFFERENCE OF TWO TERMS

Recall that when two binomials are multiplied, the product contains *four terms*. This fact is illustrated in the diagram below and is also shown by using the distributive property.

$$(a + b)(c + d) = a(c + d) + b(c + d)$$
$$= ac + ad + bc + bd$$

	a	$+$	b
c	ac		bc
$+$			
d	ad		bd

If two of the four terms of the product are similar, the similar terms can be combined so that the product is a trinomial. For example:

$$(x + 2)(x + 3) = x(x + 3) + 2(x + 3)$$
$$= x^2 + 3x + 2x + 6$$
$$= x^2 + 5x + 6$$

There is, however, a special case in which the sum of two terms is multiplied by the difference of the *same* two terms. In this case, the sum of the two middle terms of the product is 0, and the product is a binomial, as in the following examples:

$$(a + 4)(a - 4) = a(a - 4) + 4(a - 4)$$
$$= a^2 - 4a + 4a - 16$$
$$= a^2 - 16$$

$$(3x^2 - 5y)(3x^2 + 5y) = 3x^2(3x^2 + 5y) - 5y(3x^2 + 5y)$$
$$= 9x^4 + 15x^2y - 15x^2y - 25y^2$$
$$= 9x^4 - 25y^2$$

These two examples illustrate the following procedure, which enables us to find the product of the sum and difference of the same two terms mentally.

Procedure

To multiply the sum of two terms by the difference of the same two terms:

1. Square the first term.

2. From this result, subtract the square of the second term.
$$(a + b)(a - b) = a^2 - b^2$$

EXAMPLE 1

Find each product.

	Think	*Write*
a. $(y + 7)(y - 7)$	$(y)^2 - (7)^2$	$y^2 - 49$
b. $(3a - 4b)(3a + 4b)$	$(3a)^2 - (4b)^2$	$9a^2 - 16b^2$

EXERCISES

Writing About Mathematics

1. Rose said that the product of two binomials is a binomial only when the two binomials are the sum and the difference of the same two terms. Miranda said that that cannot be true because $(5a + 10)(a - 2) = 5a^2 - 20$, a binomial. Show that Rose is correct by writing the factors in the example that Miranda gave in another way.

2. Ali wrote the product $(x + 2)^2(x - 2)^2$ as $(x^2 - 4)^2$. Do you agree with Ali? Explain why or why not.

Developing Skills

In 3–17, find each product.

3. $(x + 8)(x - 8)$ **4.** $(y + 10)(y - 10)$ **5.** $(n - 9)(n + 9)$

6. $(12 + a)(12 - a)$ **7.** $(c + d)(c - d)$ **8.** $(3x + 1)(3x - 1)$

9. $(8x + 3y)(8x - 3y)$ **10.** $(x^2 + 8)(x^2 - 8)$ **11.** $(3 - 5y^3)(3 + 5y^3)$

12. $\left(a + \frac{1}{2}\right)\left(a - \frac{1}{2}\right)$ **13.** $(r + 0.5)(r - 0.5)$ **14.** $(0.3 + m)(0.3 - m)$

15. $(a + 5)(a - 5)(a^2 + 25)$ **16.** $(x - 3)(x + 3)(x^2 + 9)$ **17.** $(a + b)(a - b)(a^2 + b^2)$

Applying Skills

In 18–21, express the area of each rectangle whose length l and width w are given.

18. $l = x + 7$,   **19.** $l = 2x + 3$,   **20.** $l = c + d$,   **21.** $l = 2a + 3b$,

     $w = x - 7$ $w = 2x - 3$ $w = c - d$ $w = 2a - 3b$

In 22–25, find each product mentally by thinking of the factors as the sum and difference of the same two numbers.

22. $(27)(33) = (30 - 3)(30 + 3)$ **23.** $(52)(48) = (50 + 2)(50 - 2)$

24. $(65)(75)$ **25.** $(19)(21)$

11-5 FACTORING THE DIFFERENCE OF TWO PERFECT SQUARES

An expression of the form $a^2 - b^2$ is called a **difference of two perfect squares**. Factoring the difference of two perfect squares is the reverse of multiplying the sum of two terms by the difference of the same two terms. Since the product $(a + b)(a - b)$ is $a^2 - b^2$, the factors of $a^2 - b^2$ are $(a + b)$ and $(a - b)$. Therefore:

$$a^2 - b^2 = (a + b)(a - b)$$

Procedure

To factor a binomial that is a difference of two perfect squares:

1. Express each of its terms as the square of a monomial.

2. Apply the rule $a^2 - b^2 = (a + b)(a - b)$.

Remember that a monomial is a square if and only if its numerical coefficient is a square and the exponent of each of its variables is an even number.

EXAMPLE I

Factor each polynomial.

	Think	*Write*
a. $r^2 - 1$	$(r)^2 - (1)^2$	$(r + 1)(r - 1)$
b. $25x^2 - \frac{1}{49}y^2$	$(5x)^2 - \left(\frac{1}{7}y\right)^2$	$\left(5x + \frac{1}{7}y\right)\left(5x - \frac{1}{7}y\right)$
c. $0.04 - c^6d^4$	$(0.2)^2 - (c^3d^2)^2$	$(0.2 + c^3d^2)(0.2 - c^3d^2)$

EXAMPLE 2

Express $x^2 - 100$ as the product of two binomials.

Solution Since $x^2 - 100$ is a difference of two squares, the square of x and the square of 10, the factors of $x^2 - 100$ are $(x + 10)$ and $(x - 10)$.

Answer $(x + 10)(x - 10)$

EXERCISES

Writing About Mathematics

1. If 391 can be written as $400 - 9$, find two factors of 391 without using a calculator or pencil and paper. Explain how you found your answer.

2. Does $5a^2 - 45$ have two binomial factors? Explain your answer.

Developing Skills

In 3–18, factor each binomial.

3. $a^2 - 4$ **4.** $c^2 - 100$ **5.** $9 - x^2$ **6.** $144 - c^2$

7. $16a^2 - b^2$ **8.** $25m^2 - n^2$ **9.** $d^2 - 4c^2$ **10.** $r^4 - 9$

11. $25 - s^4$ **12.** $100x^2 - 81y^2$ **13.** $w^2 - \frac{1}{64}$ **14.** $x^2 - 0.64$

15. $0.04 - 49r^2$ **16.** $0.16y^2 - 9$ **17.** $0.81 - y^2$ **18.** $81m^4 - 49$

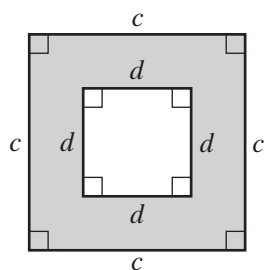### Applying Skills

In 19–23, each given polynomial represents the area of a rectangle. Express the area as the product of two binomials.

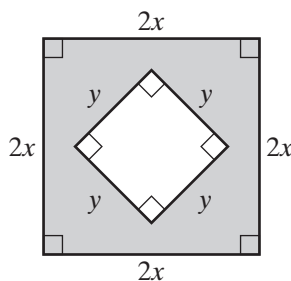**19.** $x^2 - 4$ **20.** $y^2 - 9$ **21.** $t^2 - 49$ **22.** $t^4 - 64$ **23.** $4x^2 - y^2$

In 24–26, express the area of each shaded region as: **a.** the difference of the areas shown, and **b.** the product of two binomials.
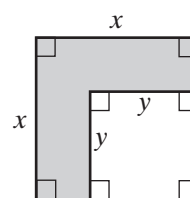
24.

25.

26.

In 27 and 28, express the area of each shaded region as the product of two binomials.

27.

28.
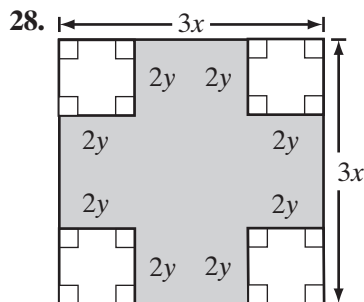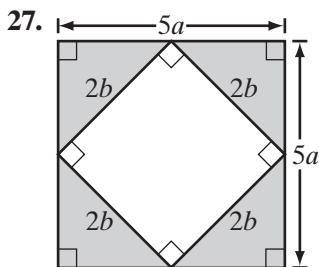
We have used the "FOIL" method from Section 5-4 to multiply two binomials of the form $ax + b$ and $cx + d$. The pattern of the multiplication in the following example shows us how to find the product of two binomials *mentally*.

$$(2x - 3)(4x + 5) = 2x(4x) + 2x(5) - 3(4x) - 3(5)$$
$$= 8x^2 + 10x - 12x - 15$$
$$= 8x^2 - 2x - 15$$

Examine the trinomial $8x^2 - 2x - 15$ and note the following:

1. The first term of the trinomial is the product of the first terms of the binomials:

$$+8x^2$$
$$(2x - 3)(4x + 5) \rightarrow 2x(4x) = 8x^2$$

2. The middle term of the trinomial is the product of the outer terms *plus* the product of the inner terms of the binomials:

$$-12x \qquad \textit{Think}$$
$$(2x - 3)(4x + 5) \rightarrow (-12x) + (+10x) = -2x$$
$$+10x$$

3. The last term of the trinomial is the product of the last terms of the binomials:

$$(2x - 3)(4x + 5) \rightarrow (-3)(+5) = -15$$
$$-15$$

Procedure

To find the product of two binomials ($ax + b$) and ($cx + d$) where *a*, *b*, *c*, and *d* have numerical values:

1. Multiply the first terms of the binomials.

2. Multiply the first term of each binomial by the last term of the other binomial (the outer and inner terms), and add these products.

3. Multiply the last terms of the binomials.

4. Write the results of steps 1, 2, and 3 as a sum.

EXAMPLE 1

Write the product $(x - 5)(x - 7)$ as a trinomial.

Solution

$(x - 5)(x - 7)$

1. $(x)(x) = x^2$
2. $(x)(-7) + (-5)(x) = (-7x) + (-5x) = -12x$
3. $(-5)(-7) = 35$
4. $x^2 - 12x + 35$ *Answer*

EXAMPLE 2

Write the product $(3y - 8)(4y + 3)$ as a trinomial.

Solution

$(3y - 8)(4y + 3)$

1. $(3y)(4y) = 12y^2$
2. $(3y)(3) + (-8)(4y) = 9y + (-32y) = -23y$
3. $(-8)(3) = -24$
4. $12y^2 - 23y - 24$ *Answer*

EXAMPLE 3

Write the product $(z + 4)^2$ as a trinomial.

Solution $(z + 4)^2 = (z + 4)(z + 4) = z^2 + 4z + 4z + 16 = z^2 + 8z + 16$ *Answer*

A trinomial, such as $z^2 + 8z + 16$, that is the square of a binomial is called a **perfect square trinomial**. Note that the first and last terms are the squares of the terms of the binomial, and the middle term is twice the product of the terms of the binomial.

EXERCISES

Writing About Mathematics

1. What is a shorter procedure for finding the product of binomials of the form $(ax + b)$ and $(ax - b)$?

2. Which part of the procedure for multiplying two binomials of the form $(ax + b)$ and $(cx + d)$ does not apply to the product of binomials such as $(3x + 5)$ and $(2y + 7)$? Explain your answer.

Developing Skills

In 3–29, perform each indicated operation mentally.

3. $(x + 5)(x + 3)$

4. $(6 + d)(3 + d)$

5. $(x - 10)(x - 5)$

6. $(8 - c)(3 - c)$

7. $(x + 7)(x - 2)$

8. $(n - 20)(n + 3)$

9. $(5 - t)(9 + t)$

10. $(3x + 2)(x + 5)$

11. $(c - 5)(3c - 1)$

12. $(y + 8)^2$

13. $(a - 4)^2$

14. $(2x + 1)^2$

15. $(3x - 2)^2$

16. $(7x + 3)(2x - 1)$

17. $(2y + 3)(3y + 2)$

18. $(3x + 4)^2$

19. $(2x - 5)^2$

20. $(3t - 2)(4t + 7)$

21. $(5y - 4)(5y - 4)$

22. $(2t + 3)(5t + 1)$

23. $(2a - b)(2a - 3b)$

24. $(5x + 7y)(3x - 4y)$

25. $(2c - 3d)(5c - 2d)$

26. $(a + b)^2$

27. $(a + b)(2a - 3)$

28. $(6t - 1)(4t + z)$

29. $(9y - w)(9y + 3w)$

Applying Skills

30. Write the trinomial that represents the area of a rectangle whose length and width are:

    a. $(x + 5)$ and $(x + 4)$ **b.** $(2x + 3)$ and $(x - 1)$

31. Write the perfect square trinomial that represents the area of a square whose sides are:

    a. $(x + 6)$ **b.** $(x - 2)$ **c.** $(2x + 1)$ **d.** $(3x - 2)$

32. The length of a garden is 3 feet longer than twice the width.

    a. If the width of the garden is represented by x, represent the length of the garden in terms of x.

    b. If the width of the garden is increased by 5 feet, represent the new width in terms of x.

    c. If the length of the enlarged garden is also increased so that it is 3 feet longer than twice the new width, represent the length of the enlarged garden in terms of x.

    d. Express the area of the enlarged garden in terms of x.

11-7 FACTORING TRINOMIALS

We have learned that $(x + 3)(x + 5) = x^2 + 8x + 15$. Therefore, factors of $x^2 + 8x + 15$ are $(x + 3)$ and $(x + 5)$. Factoring a trinomial of the form $ax^2 + bx + c$ is the reverse of multiplying binomials of the form $(dx + e)$ and $(fx + g)$. When we factor a trinomial of this form, we use combinations of factors of the first and last terms. We list the possible pairs of factors, then test the pairs, one by one, until we find the pair that gives the correct middle term.

For example, let us factor $x^2 + 7x + 10$:

1. The product of the first terms of the binomials must be x^2. Therefore, for each first term, we use x. We write:

$$x^2 + 7x + 10 = (x \quad)(x \quad)$$

2. Since the product of the last terms of the binomials must be $+10$, these last terms must be either both positive or both negative. The pairs of integers whose products is $+10$ are:

$$(+1)(+10) \qquad (+5)(+2) \qquad (-1)(-10) \qquad (-2)(-5)$$

3. From the products obtained in steps 1 and 2, we see that the possible pairs of factors are:

$$(x + 10)(x + 1) \qquad (x + 5)(x + 2)$$
$$(x - 10)(x - 1) \qquad (x - 5)(x - 2)$$

4. Now, we test each pair of factors:

$(x + 10)(x + 1)$ is not correct because the middle term is $+11x$, not $+7x$.

$(x + 5)(x + 2)$ is correct because the middle term is $+7x$.

$$\overset{+10x}{(x + 10)(x + 1)} \to 10x + 1x = 11x \quad \textbf{✗}$$
$$\underset{+1x}{}$$

$$\overset{+5x}{(x + 5)(x + 2)} \to 2x + 5x = 7x \quad \textbf{✔}$$
$$\underset{+2x}{}$$

Neither of the remaining pairs of factors is correct because each would result in a negative middle term.

5. The factors of $x^2 + 7x + 10$ are $(x + 5)$ and $(x + 2)$.

Observe that in this trinomial, the first term, x^2, is positive and must have factors with the same signs. We usually choose positive factors of the first terms. The last term, $+10$, is positive and must have factors with the same signs. Since the middle term of the trinomial is also *positive*, the last terms of both binomial factors must be *positive* ($+5$ and $+2$).

In this example, if we had chosen $-x$ times $-x$ as the factors of x^2, the factors of $x^2 + 7x + 10$ would have been written as $(-x - 5)$ and $(-x - 2)$. Every trinomial that has two binomial factors also has the opposites of these binomials as factors. Usually, however, we write only the pair of factors whose first terms have positive coefficients as the factors of a trinomial.

Procedure

To factor a trinomial of the form $ax^2 + bx + c$, find two binomials that have the following characteristics:

1. The product of the first terms of the binomials is equal to the first term of the trinomial (ax^2).

2. The product of the last terms of the binomials is equal to the last term of the trinomial (c).

3. When the first term of each binomial is multiplied by the last term of the other binomial and the sum of these products is found, the result is equal to the middle term of the trinomial.

EXAMPLE 1

Factor $y^2 - 8y + 12$.

Solution (1) The product of the first terms of the binomials must be y^2. Therefore, for each first term, we use y. We write:

$$y^2 - 8y + 12 = (y \quad)(y \quad)$$

(2) Since the product of the last terms of the binomials must be +12, these last terms must be either both positive or both negative. The pairs of integers whose product is +12 are:

$$(+1)(+12) \qquad (+2)(+6) \qquad (+3)(+4)$$
$$(-1)(-12) \qquad (-2)(-6) \qquad (-3)(-4)$$

(3) The possible factors are:

$$(y + 1)(y + 12) \qquad (y + 2)(y + 6) \qquad (y + 3)(y + 4)$$
$$(y - 1)(y - 12) \qquad (y - 2)(y - 6) \qquad (y - 3)(y - 4)$$

(4) When we find the middle term in each of the trinomial products, we see that only for the factors $(y - 6)(y - 2)$ is the middle term $-8y$:

$$\overset{-6y}{\underline{(y - 6)(y - 2)}} \to -2y - 6y = -8y$$
$$\underset{-2y}{}$$

Answer $y^2 - 8y + 12 = (y - 6)(y - 2)$

When the first and last terms are both positive (y^2 and +12 in this example) and the middle term of the trinomial is *negative*, the last terms of both binomial factors must be *negative* (-6 and -2 in this example).

EXAMPLE 2

Factor: **a.** $c^2 + 5c - 6$ **b.** $c^2 - 5c + 6$

Solution **a.** (1) The product of the first terms of the binomials must be c^2. Therefore, for each first term, we use c. We write:

$$c^2 + 5c - 6 = (c \quad)(c \quad)$$

(2) Since the product of the last terms of the binomials must be –6, one of these last terms must be positive and the other negative. The pairs of integers whose product is –6 are:

$$(+1)(-6) \qquad\qquad (+3)(-2)$$
$$(-1)(+6) \qquad\qquad (-3)(+2)$$

(3) The possible factors are:

$$(c + 1)(c - 6) \qquad\qquad (c + 3)(c - 2)$$
$$(c - 1)(c + 6) \qquad\qquad (c - 3)(c + 2)$$

(4) When we find the middle term in each of the trinomial products, we see that only for the factors $(c - 1)(c + 6)$ is the middle term $+5c$:

$$\overset{-1c}{(c - 1)(c + 6)} \to 6c - 1c = 5c$$
$$\underset{+6c}{}$$

b. (1) As in the solution to part **a**, we write:

$$c^2 - 5c + 6 = (c \quad)(c \quad)$$

(2) Since the product of the last terms of the binomials must be $+6$, these last terms must be either both positive or both negative. The pairs of integers whose product is $+6$ are:

$$(+1)(+6) \qquad\qquad (+3)(+2)$$
$$(-1)(-6) \qquad\qquad (-3)(-2)$$

(3) The possible factors are:

$$(c + 1)(c + 6) \qquad\qquad (c + 3)(c + 2)$$
$$(c - 1)(c - 6) \qquad\qquad (c - 3)(c - 2)$$

(4) When we find the middle term in each of the trinomial products, we see that only for the factors $(c - 3)(c - 2)$ is the middle term $-5c$.

$$\overset{-3c}{(c - 3)(c - 2)} \to -2c - 3c = -5c$$
$$\underset{-2c}{}$$

Answer **a.** $c^2 + 5c - 6 = (c - 1)(c + 6)$ **b.** $c^2 - 5c + 6 = (c - 3)(c - 2)$

EXAMPLE 3

Factor $2x^2 - 7x - 15$.

Solution (1) Since the product of the first terms of the binomials must be $2x^2$, we use $2x$ and x as the first terms. We write:

$$2x^2 - 7x - 15 = (2x \quad)(x \quad)$$

(2) Since the product of the last terms of the binomials must be -15, one of these last terms must be positive and the other negative. The pairs of integers whose product is -15 are:

$$(+1)(-15) \qquad\qquad (+3)(-5)$$
$$(-1)(+15) \qquad\qquad (-3)(+5)$$

(3) These four pairs of integers will form eight pairs of binomial factors since there are two ways in which the first terms can be arranged. Note how $(2x + 1)(x - 15)$ is not the same product as $(2x - 15)(x + 1)$. The possible pairs of factors are:

$(2x + 1)(x - 15)$ $(2x + 3)(x - 5)$ $(2x - 1)(x + 15)$ $(2x - 3)(x + 5)$
$(2x + 15)(x - 1)$ $(2x + 5)(x - 3)$ $(2x - 15)(x + 1)$ $(2x - 5)(x + 3)$

(4) When we find the middle term in each of the trinomial products, we see that only for the factors $(2x + 3)(x - 5)$ is the middle term $-7x$:

$$\overset{+3x}{(2x + 3)(x - 5)} \to 3x - 10x = -7x$$
$$\underset{-10x}{}$$

Answer $2x^2 - 7x - 15 = (2x + 3)(x - 5)$

In factoring a trinomial of the form $ax^2 + bx + c$, when a is a positive integer ($a > 0$):

1. The coefficients of the first terms of the binomial factors are usually written as positive integers.

2. If the last term, c, is positive, the last terms of the binomial factors must be either both positive (when the middle term, b, is positive), or both negative (when the middle term, b, is negative).

3. If the last term, c, is negative, one of the last terms of the binomial factors must be positive and the other negative.

EXERCISES

Writing About Mathematics

1. Alicia said that the factors of $x^2 + bx + c$ are $(x + d)(x + e)$ if $c = de$ and $b = d + e$. Do you agree with Alicia? Explain why or why not.

2. Explain why there is no positive integral value of c for which $x^2 + x + c$ has two binomial factors.

Developing Skills

In 3–33, factor each trinomial.

3. $a^2 + 3a + 2$	**4.** $c^2 + 6c + 5$	**5.** $x^2 + 8x + 7$	**6.** $x^2 - 11x + 10$
7. $y^2 - 6y + 8$	**8.** $y^2 + 6y + 8$	**9.** $y^2 - 9y + 8$	**10.** $y^2 + 9y + 8$
11. $y^2 - 2y - 8$	**12.** $y^2 + 2y - 8$	**13.** $y^2 - 7y - 8$	**14.** $y^2 + 7y - 8$
15. $x^2 + 11x + 24$	**16.** $a^2 - 11a + 18$	**17.** $z^2 + 10z + 25$	**18.** $x^2 - 5x + 6$
19. $x^2 - 10x + 24$	**20.** $x^2 - x - 2$	**21.** $x^2 - 6x - 7$	**22.** $y^2 + 4y - 5$
23. $c^2 + 2c - 35$	**24.** $x^2 - 7x - 18$	**25.** $z^2 + 9z - 36$	**26.** $2x^2 + 5x + 2$
27. $3x^2 + 10x + 8$	**28.** $16x^2 + 8x + 1$	**29.** $2x^2 + x - 3$	**30.** $4x^2 - 12x + 5$
31. $10a^2 - 9a + 2$	**32.** $3a^2 - 7ab + 2b^2$	**33.** $4x^2 - 5xy - 6y^2$	

Applying Skills

In 34–36, each trinomial represents the area of a rectangle. In each case, find two binomials that could be expressions for the dimensions of the rectangle.

34. $x^2 + 10x + 9$ **35.** $x^2 + 9x + 20$ **36.** $3x^2 + 14x + 15$

In 37–39, each trinomial represents the area of a square. In each case, find a binomial that could be an expression for the measure of each side of the square.

37. $x^2 + 10x + 25$ **38.** $81x^2 + 18x + 1$ **39.** $4x^2 + 12x + 9$

11-8 FACTORING A POLYNOMIAL COMPLETELY

Some polynomials, such as $x^2 + 4$ and $x^2 + x + 1$, cannot be factored into other polynomials with integral coefficients. We say that these polynomials are **prime over the set of integers**.

Factoring a polynomial completely means finding the prime factors of the polynomial over a designated set of numbers. In this book, we will consider a polynomial factored when it is written as a product of monomials or prime polynomials over the set of integers.

Procedure

To factor a polynomial completely:

1. Look for the greatest common factor. If it is greater than 1, factor the given polynomial into a monomial times a polynomial.

2. Examine the polynomial factor or the given polynomial if it has no common factor (the greatest common factor is 1). Then:

   • Factor any trinomial into the product of two binomials if possible.

   • Factor any binomials that are the difference of two perfect squares as such.

3. Write the given polynomial as the product of all of the factors. Make certain that all factors except the monomial factor are prime polynomials.

EXAMPLE 1

Factor $by^2 - 4b$.

Solution   *How to Proceed*

(1) Find the greatest common factor of the terms: $by^2 - 4b = b(y^2 - 4)$

(2) Factor the difference of two squares: $by^2 - 4b = b(y + 2)(y - 2)$ *Answer*

EXAMPLE 2

Factor $3x^2 - 6x - 24$.

Solution   *How to Proceed*

(1) Find the greatest common factor of the terms: $3x^2 - 6x - 24 = 3(x^2 - 2x - 8)$

(2) Factor the trinomial: $3x^2 - 6x - 24 = 3(x - 4)(x + 2)$ *Answer*

EXAMPLE 3

Factor $4d^2 - 6d + 2$.

Solution   *How to Proceed*

(1) Find the greatest common factor of the terms: $4d^2 - 6d + 2 = 2(2d^2 - 3d + 1)$

(2) Factor the trinomial: $4d^2 - 6d + 2 = 2(2d - 1)(d - 1)$ *Answer*

EXAMPLE 4

Factor $x^4 - 16$.

Solution	*How to Proceed*	
	(1) Find the greatest common factor of the terms:	The greatest common factor is 1.
	(2) Factor the binomial as the difference of two squares:	$x^4 - 16 = (x^2 + 4)(x^2 - 4)$
	(3) Factor the difference of two squares:	$x^4 - 16 = (x^2 + 4)(x + 2)(x - 2)$ *Answer*

EXERCISES

Writing About Mathematics

1. Greta said that since $4a^2 - a^2b^2$ is the difference of two squares, the factors are $(2a + ab)(2a - ab)$. Has Greta factored $4a^2 - a^2b^2$ into prime polynomial factors? Explain why or why not.

2. Raul said that the factors of $x^3 - 1$ are $(x + 1)(x + 1)(x - 1)$. Do you agree with Raul? Explain why or why not.

Developing Skills

In 3–29, factor each polynomial completely.

3. $2a^2 - 2b^2$

4. $4x^2 - 4$

5. $ax^2 - ay^2$

6. $st^2 - 9s$

7. $2x^2 - 32$

8. $3x^2 - 27y^2$

9. $18m^2 - 8$

10. $63c^2 - 7$

11. $x^3 - 4x$

12. $z^3 - z$

13. $4a^2 - 36$

14. $x^4 - 1$

15. $y^4 - 81$

16. $\pi c^2 - \pi d^2$

17. $3x^2 + 6x + 3$

18. $4r^2 - 4r - 48$

19. $x^3 + 7x^2 + 10x$

20. $4x^2 - 6x - 4$

21. $d^3 - 8d^2 + 16d$

22. $2ax^2 - 2ax - 12a$

23. $16x^2 - x^2y^4$

24. $a^4 - 10a^2 + 9$

25. $y^4 - 13y^2 + 36$

26. $5x^4 + 10x^2 + 5$

27. $2a^2b + 7ab + 3b$

28. $16x^2 - 16x + 4$

29. $25x^2 + 100xy + 100y^2$

Applying Skills

30. The volume of a rectangular solid is represented by $12a^3 - 5a^2b - 2ab^2$. Find the algebraic expressions that could represent the dimensions of the solid.

31. A rectangular solid has a square base. The volume of the solid is represented by $3m^2 + 12m + 12$.

a. Find the algebraic expression that could represent the height of the solid.

b. Find the algebraic expression that could represent the length of each side of the base.

32. A rectangular solid has a square base. The volume of the solid is represented by $10a^3 + 20a^2 + 10a$.

a. Find the algebraic expression that could represent the height of the solid.

b. Find the algebraic expression that could represent the length of each side of the base

CHAPTER SUMMARY

Factoring a number or a polynomial is the process of finding those numbers or polynomials whose product is the given number or polynomial. A **perfect square trinomial** is the square of a binomial, whereas the **difference of two perfect squares** is the product of binomials that are the sum and difference of the same two. A **prime polynomial**, like a prime number, has only two factors, 1 and itself.

To factor a polynomial completely:

1. Factor out the greatest common monomial factor if it is greater than 1.

2. Write any factor of the form $a^2 - b^2$ as $(a + b)(a - b)$.

3. Write any factor of the form $ax^2 + bx + c$ as the product of two binomial factors if possible.

4. Write the given polynomial as the product of these factors.

VOCABULARY

11-1 Product • Factoring a number • Factoring over the set of integers • Greatest common factor

11-2 Factoring a polynomial • Common monomial factor • Greatest common monomial factor • Prime polynomial

11-3 Square of a monomial

11-5 Difference of two perfect squares

11-6 Perfect square trinomial

11-8 Prime over the set of integers • Factoring a polynomial completely

REVIEW EXERCISES

1. Leroy said that $4x^2 + 16x + 12 = (2x + 2)(2x + 6) = 2(x + 1)(x + 3)$. Do you agree with Leroy? Explain why or why not.

2. Express 250 as a product of prime numbers.

3. What is the greatest common factor of $8ax$ and $4ay$?

4. What is the greatest common factor of $16a^3bc^2$ and $24a^2bc^4$?

In 5–8, square each monomial.

5. $(3g^3)^2$ **6.** $(-4x^4)^2$ **7.** $(0.2c^2y)^2$ **8.** $\left(\frac{1}{2}a^3b^5\right)^2$

In 9–14, find each product.

9. $(x - 5)(x + 9)$ **10.** $(y - 8)(y - 6)$ **11.** $(ab + 4)(ab - 4)$

12. $(3d + 1)(d - 2)$ **13.** $(2w + 1)^2$ **14.** $(2x + 3c)(x + 4c)$

In 15–29, in each case factor completely.

15. $6x + 27b$ **16.** $3y^2 + 10y$ **17.** $m^2 - 81$

18. $x^2 - 16h^2$ **19.** $x^2 - 4x - 5$ **20.** $y^2 - 9y + 14$

21. $64b^2 - 9$ **22.** $121 - k^2$ **23.** $x^2 - 8x + 16$

24. $a^2 - 7a - 30$ **25.** $x^2 - 16x + 60$ **26.** $16y^2 - 16$

27. $2x^2 + 12bx - 32b^2$ **28.** $x^4 - 1$ **29.** $3x^3 - 6x^2 - 24x$

30. Express the product $(k + 15)(k - 15)$ as a binomial.

31. Express $4ez^2(4e - z)$ as a binomial.

32. Factor completely: $60a^2 + 37a - 6$.

33. Which of the following polynomials has a factor that is a perfect square trinomial and a factor that is a perfect square?
(1) $a^2y + 10ay + 25y$ (3) $18m^2 + 24m + 8$
(2) $2ax^2 - 2ax - 12a$ (4) $c^2z^2 - 18cz^2 + 81z^2$

34. Of the four polynomials given below, explain how each is different from the others.

$x^2 - 9$ $x^2 - 2x + 1$ $x^2 - 2x - 1$ $x^3 + 5x^2 + 6x$

35. If the length and width of a rectangle are represented by $2x - 3$ and $3x - 2$, respectively, express the area of the rectangle as a trinomial.

36. Find the trinomial that represents the area of a square if the measure of a side is $8m + 1$.

37. If $9x^2 + 30x + 25$ represents the area of a square, find the binomial that represents the length of a side of the square.

38. A group of people wants to form several committees, each of which will have the same number of persons. Everyone in the group is to serve on one and only one committee. When the group tries to make 2, 3, 4, 5, or 6 committees, there is always one extra person. However, they are able to make more than 6 but fewer than 12 committees of equal size.

  a. What is the smallest possible number of persons in the group?

  b. Using the group size found in **a**, how many persons are on each committee that is formed?
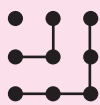
Exploration

a. Explain how the expression $(a + b)(a - b) = a^2 - b^2$ can be used to find the product of two consecutive even or two consecutive odd integers.

b. The following diagrams illustrate the formula:
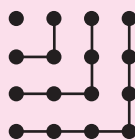
$$1 + 3 + 5 + \ldots + (2n - 3) + (2n - 1) = n^2$$

$1 + 3 = 4 = 2^2$

$1 + 3 + 5 = 9 = 3^2$

$1 + 3 + 5 + 7 = 16 = 4^2$

  (1) Explain how 3, 5, and 7 are the difference of two squares using the diagrams.
  (2) Use the formula to explain how any odd number, $2n - 1$, can be written as the difference of squares.
  [*Hint:* $1 + 3 + 5 + \ldots + (2n - 3) = (n - 1)^2$]

CUMULATIVE REVIEW   CHAPTERS 1–11

Part I

Answer all questions in this part. Each correct answer will receive 2 credits. No partial credit will be allowed.

1. If apples cost $3.75 for 3 pounds, what is the cost, at the same rate, of 7 pounds of apples?

(1) $7.25   (2) $8.25   (3) $8.50   (4) $8.75

2. When $3a^2 - 4$ is subtracted from $2a^2 - 4a$, the difference is
   (1) $-a^2 - 4a + 4$ (2) $a^2 + 4a - 4$ (3) $a^2 + 8a$ (4) $-a^2 - a$

3. The solution of the equation $x - 3(2x + 4) = 7x$ is
   (1) -6 (2) 6 (3) -1 (4) 1

4. The area of a circle is 16π square centimeters. The circumference of the circle is
   (1) 16π centimeters   (3) 4π centimeters
   (2) 8π centimeters   (4) 2π centimeters

5. The x-intercept of the line whose equation is $2x - y = 5$ is
   (1) 5 (2) -5 (3) $\frac{2}{5}$ (4) $\frac{5}{2}$

6. The factors of $x^2 - 3x - 10$ are
   (1) $(x + 5)(x - 2)$ (3) $(x - 5)(x + 2)$
   (2) $(x - 5)(x - 2)$ (4) $(x + 5)(x + 2)$

7. The fraction $\frac{2.5 \times 10^{-5}}{5.0 \times 10^{-8}}$ is equal to
   (1) 5×10^{-3} (2) 5×10^{-2} (3) 5×10^{2} (4) 5×10^{3}

8. When factored completely, $5a^3 - 45a$ is equal to
   (1) $5a(a + 3)^2$ (3) $a(5a + 9)(a - 5)$
   (2) $5(a^2 + 3)(a - 3)$ (4) $5a(a + 3)(a - 3)$

9. When $a = -3$, $-a^2 + 5a$ is equal to
   (1) 6 (2) -6 (3) 24 (4) -24

10. The graph of the equation $2x + y = 7$ is parallel to the graph of
   (1) $y = 2x + 3$ (2) $y = -2x + 5$ (3) $x + 2y = 4$ (4) $y - 2x = 2$

Part II

Answer all questions in this part. Each correct answer will receive 2 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

11. The measures of the sides of a right triangle are 32, 60, and 68. Find, to the nearest degree, the measure of the smallest angle of the triangle.

12. Solve and check: $\frac{x}{x + 8} = \frac{2}{3}$.

Part III

Answer all questions in this part. Each correct answer will receive 3 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

13. The width of Mattie's rectangular garden is 10 feet less than the length. She bought 25 yards of fencing and used all but 7 feet to enclose the garden.

    a. Write an equation or a system of equations that can be used to find the dimensions, in feet, of the garden.

    b. What are the dimensions, in feet, of the garden?

14. The population of a small town at the beginning of the year was 7,000. Records show that during the year there were 5 births, 7 deaths, 28 new people moved into town, and 12 residents moved out. What was the percent of increase or decrease in the town population?

Part IV

Answer all questions in this part. Each correct answer will receive 4 credits. Clearly indicate the necessary steps, including appropriate formula substitutions, diagrams, graphs, charts, etc. For all questions in this part, a correct numerical answer with no work shown will receive only 1 credit.

15. Admission to a museum is $5.00 for adults and $3.00 for children. In one day $2,400 was collected for 620 paid admissions. How many adults and how many children paid to visit the museum that day?

16. a. Write an equation of a line whose slope is –2 and whose y-intercept is 5.

    b. Sketch the graph of the equation written in **a**.

    c. Are $(-1, 3)$ the coordinates of a point on the graph of the equation written in **a**?

    d. If $(k, 1)$ are the coordinates of a point on the graph of the equation written in **a**, what is the value of k?